# GCSE Algebra Rules and Tools

## Table of Contents

# Symbols and conventions

| Symbol with example | Meaning | Notes and associated words |
|---|---|---|
| 3 + 4 | Adding, addition | Sum, total |
| 11 – 6 | Subtracting, subtraction | Difference |
| 2 × 5 | Multiply, multiplication | Product |
| 15 ÷ 5 | Divide, division | Quotient (rare!) |
| 15.239261 ≈ 15 | Approximately, approximation | Estimate, round off, round up |
| 5 > 3 | Greater than | |
| 2 < 1000 | Less than | |
| 21 ≥ 17 | Greater than or equal to | |
| 2 ≤ 10 | Less than or equal to | |
| 2 ≠ 7 | Not equal to | |
| ≡ | Equivalent, equivalence | Used mainly in algebra |
| $2^{10}$ | Power, index, indices | The 10 is small and above the line of writing |
| 15ºC | Degrees Celsius | The little circle is different to a zero. If temperature then there has to be a letter after the little circle to say which temperature scale.<br>ºC is degrees Celsius,<br>ºF is degrees Fahrenheit,<br>ºK is degrees Kelvin (degrees above absolute zero, H level) |
| 45º | Degrees | Without a letter after it, the little circle stands for the degrees you measure with a protractor, the size of an angle |

# Algebra

Algebra is a way of talking systematically about the process of calculating.

Symbols are used so that you can talk about the method of a calculation without working with actual examples.

We will start with a formula that is used commonly as a part of GCSE exam questions.

## Speed distance and time

**Example**: Suppose the coach travels at a steady 50 miles per hour for three hours up the motorway. How far have you travelled?

Each hour is 50 miles, so three hours is 50 × 3 = 150 miles.

### *The formula(s)*

From the above example, you can write the formula D = S × T

Where D stands for distance, S stands for speed and T stands for time.

**Example**: If you were travelling at a steady 30 miles per hour, how many hours would you need to travel 120 miles?

120 ÷ 30 = 4 hours

From this example, you can write the formula $T=\frac{D}{S}$

**Example**: If it took 2 hours to travel 35 miles one very busy morning, what was the average speed of your journey?

35 ÷ 2 = 17.5 miles per hour

From this example, you can write the formula $S=\frac{D}{T}$

### *Decimal hours*

If your speed is in miles per hour, then the distance must be measured in miles and the time must be measured in hours.

Remember that 2 hours and 45 minutes is 2.75 hours NOT 2.45 hours.

To convert minutes into decimal hours, you calculate $\frac{minutes}{60}$

so 24 minutes is $\frac{24}{60}=0.4$ hours

**Example**: Fred drives a white van. A trip from Birmingham to Walsall and back, a distance of 30 km took 1 hour and 45 minutes. Calculate Fred's average speed in km/h.

**Stage 1**: 1h 45 m = 1 + $\frac{45}{60}$ = 1.75 hours.

**Stage 2**: Using $S=\frac{D}{T}$ you find that $S=\frac{30}{1.75}=17.14$ km/h

To convert from decimal hours back to hours and minutes, just multiply the decimal fraction part by 60

**Example**: write 3.48 hours in hours and minutes

**Stage 1**: First find 0.48 × 60 = 28.8 minutes ≈ 29 minutes

**Stage 2**: Then add on the whole hours and write it out: 3h 29m

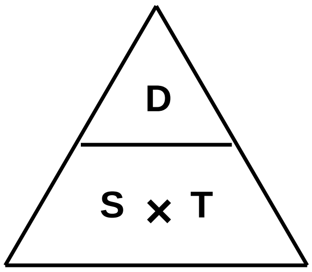
**Example**: How long will it take Nahida to drive 125 miles at an average speed of 40 miles per hour?

**Stage 1**: use the $T=\frac{D}{S}$ formula $T=\frac{125}{40}=3.125$ so it will take 3.125 decimal hours

**Stage 2**: convert 3.125 decimal hours to hours and minutes

Minutes: 0.125 × 60 = 7.5 minutes ≈ 8 minutes + 3 hours so 3h 8m

## *Formula triangle*

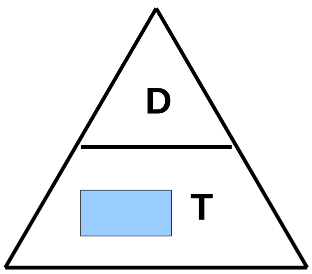



You can remember what calculation to do by using the formula triangle above

To use the triangle, cover up the letter you are trying to find, and the other two letters tell you the calculation to use.

**Example**: Aaron has sailed 250 miles in 12 hours. What is his average speed in miles per hour?

**Stage 1**: We want to find Aaron's speed, so we cover up the S in the formula triangle




**Stage 2**: The remaining letters say to divide the distance by the time $\frac{250}{12}=20.83...$ so roughly 21 miles per hour.

**Challenge**: find an exercise about speed, distance and time in a GCSE Maths textbook and check your answers. Try some questions in metres per second as well as the ones in miles or kilometres per hour.

# BIDMAS: Sequence of Operations

**Example**: 3 + 4 × 2 = 3 + 8 = 11, NOT 14

When you have more than one mathematical operation (+ – × ÷ ) in a calculation you don't read the calculation from left to right.

Multiplying and dividing are calculated before the adding and subtracting.

It is possible to force adding to be completed before other operations by putting the addition in brackets

**Example**: (3 + 4) × 2 = 7 × 2 = 14

The word BIDMAS is used to remember the sequence of operations.

**Example**: Evaluate (99 – 87) ÷ $2^2$ + 2 × 5 – 7

Work step by step down the BIDMAS order of operations

| | | |
|---|---|---|
| **B**rackets | 12 ÷ $2^2$ + 2 × 5 – 7 | We calculated the value of the bracket |
| **I**ndex | 12 ÷ 4 + 2 × 5 – 7 | Now we have replaced the $2^2$ with its value |
| **D**ivide | 3 + 2 × 5 – 7 | 12 ÷ 4 = 3 so we replace the 12 ÷ 4 with 3 |
| **M**ultiply | 3 + 10 – 7 | 2 × 5 = 10 so replace the 2 × 5 with 10 |
| **A**dd | 13 | Add |
| **S**ubtract | 6 | Subtract |

## *Fraction bars mean brackets round top and bottom*

**Example**: evaluate $\frac{999-993}{47-45}$

**Stage 1**: The fraction bar means divide, but you have to work out the top and bottom first, so treat the top and bottom of any fraction bar expression as if they were in brackets

$$\frac{999-993}{47-45}=(999-993)\div(47-45)$$

**Stage 2**: Use BIDMAS to calculate $(999-993)\div(47-45)=6\div 2=3$

**Challenge**: Google “BIDMAS Calculations” and work through the Maths is Fun page and complete the questions at the bottom of the page. The Maths Is Fun page uses BODMAS as the word to remember the sequence of operations. The end result is just the same!

# Estimating

Estimating the result of a complex calculation draws on rounding to significant figures and on BIDMAS.

**Example**: Estimate the value of $\frac{21.15 \times 4.97^2}{0.04899}$

**Stage 1**: Round all the figures to one significant figure $\frac{20 \times 5^2}{0.05}$

**Stage 2**: BIDMAS on the rounded version $\frac{20 \times 5^2}{0.05} = \frac{20 \times 25}{0.05} = \frac{500}{0.05}$

**Stage 3**: Recall dividing by decimals $\frac{500}{0.05} = \frac{50000}{5} = 10000$

(multiply top and bottom by 100 so you are dividing by a whole number)

Notice how dividing by a number much smaller than 1 increases the size of the answer.

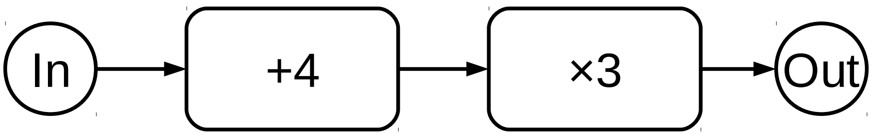
# Function Machines

Function machines are diagrams that look like a flowchart that help you understand the process of performing a calculation.

When I teach algebra, I use function machines as a stepping stone to the 'traditional' use of symbols.

## *Getting the output from the input*

**Example**: Find the output of the function machine below when the input is 2.




**Stage 1**: Imagine feeding the value 2 into the 'input' and then working your way along the flowchart until you reach the 'output' box.

**Stage 2**: The first box gives 2 + 4 = 6

**Stage 3**: The second box gives 6 × 3 = 18 so the 'output' is 18.

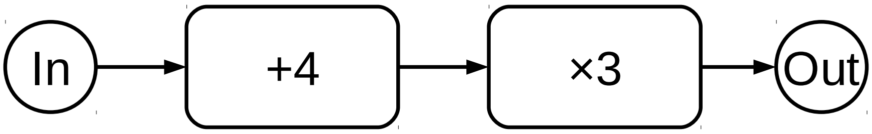
**Example**: Complete the table below using the function machine

| **Input** | **Box 1** | **Box 2** | **Output** |
|---|---|---|---|
| – 5 | – 5 + 4 = –1 | –1 × 3 = –3 | –3 |
| 6 | 6 + 4 = 10 | 10 × 3 = 30 | 30 |
| x | $x+4$ | $(x+4) \times 3$ | $3(x+4)$ |

The last row shows how you can use BIDMAS to write the calculation in a function machine using 'traditional' algebraic notation. See below

## *Turning a function machine round*

**Example**: Below is a function machine




Make a new function machine that takes the output from this one and generates the input.

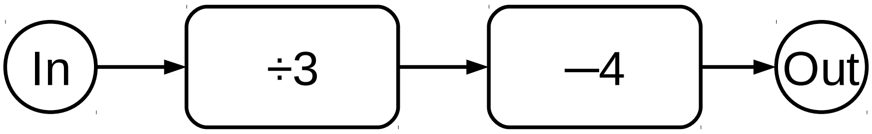
**Stage 1**: Change each operation for its opposite,

so +4 becomes –4 and ×3 becomes ÷3

**Stage 2**: Reverse the order of the boxes,

so ÷3 first then –4

The new function machine looks like this...




**Check**: You know that the original function machine gave output 18 for input 2

So if we input 18, this new function machine should give output 2,

18 ÷3 = 6 –4 = 2 so it works!

## *Making function machines from situations*

**Example**: The Big Delivery Company charge £1.75 per mile for delivering within the town, plus a £30 call out charge.

Draw up a function machine to calculate the total cost of a delivery.

**Stage 1**: Say what the input is and what the output should be

Input is the number of miles for the delivery, output is the cost of the delivery

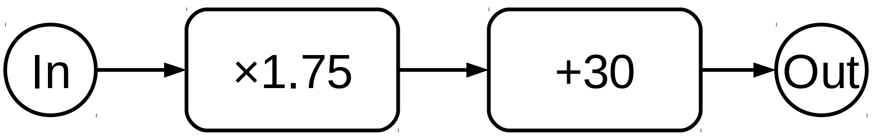
**Stage 2**: Break the calculation down into stages; use a real input.

Suppose the delivery is 4 miles. The cost will be 4 × £1.75 then add £30

**Stage 3**: Each stage needs a box

The first box will be × 1.75 and the second box will be +30

So the function machine looks like this




**Challenge**: check the function machine above with input 4 miles. Turn the function machine around to see how far you can go with £47.50

# Expressions, terms and conventions

In algebra, we use letters to stand for numbers and then write the operations using the familiar signs.

An <u>expression</u> looks like this: $3x+4y-2x^2$ or this $\frac{2x+7y}{3y^2x^2}$ or this $3(4x+7)-2(2x+5)$ basically a collection of symbols and operations.

A <u>term</u> is part of an expression, so the expression $3x+4y-2x^2$ is made up of the terms $+3x$ , $+4y$ and $-2x^2$ .

As you can see, each term has a sign, a number and a letter. The letter might have a power.

What does $+3x$ mean? It means “multiply the value of the letter x by 3 then add it on”.

The symbol $x$ is also a term. As there is no sign, we assume adding, as there is no number, we assume to just add the value of $x$ .

You *could* write the term $x$ as $+1x$ but the convention is to not write the 1 in the term.

The next section will show you how to substitute specific values of the various letters into expressions and calculate numerical answers. This will help 'pin down' what expressions are and how algebra works.

# Substituting into expressions and formulas

**Example**: Calculate the value of $\frac{3p+4q}{2}$ when $p=2$ and $q=3$

**Stage 1**: <u>Substitute</u> the letters for their values $\frac{3p+4q}{2}=\frac{3\times 2+4\times 3}{2}$

**Stage 2**: Calculate the answer using BIDMAS

$$\frac{3p+4q}{2}=\frac{3\times 2+4\times 3}{2}=\frac{6+12}{2}=\frac{18}{2}=9$$

Negative numbers and fractions can feature in substitutions as well

**Example**: The cost C of manufacturing a small part is given by the formula

$C=0.05n+550$ where n is the number of the small parts that are made in one batch.

Calculate the cost of making 10 000 of the small parts

$C=0.05n+550=0.05\times 10000+550=500+550=1050$ so the batch costs £1050.

**Example**:The formula $F=1.8\mathrm{C}+32$ can be used to calculate the temperature in degrees Fahrenheit (F) from the temperature in degrees Celsius (C).

The temperature is –14°C. Calculate the temperature in degrees Fahrenheit.

**Stage 1**: Replace the letters with their values $F=1.8C+32=1.8\times -14+32$

**Stage 2**: Apply BIDMAS rules to calculate the answer

$1.8\times -14+32=-25.2+32=6.8$ so the temperature is 6.8°F

**Challenge**: Find a substitution exercise in a GCSE textbook and check your answers!

## Making expressions from words

**Example**: Mkose buys paving slabs for £4.50 each.

Write an expression for the cost of n slabs.

**Stage 1**: Imagine a number of slabs, like 50. Write the calculation for that you would do for that number of slabs: 4.5 × 50

**Stage 2**: Replace the number you made up with n so the expression becomes 4.5n

**Example**: Efilda cooks a chicken as follows: Allow 30 minutes plus 20 minutes per pound

Write a formula for the cooking time T in terms of the weight W of the chicken.

**Stage 1**: Imagine a weight for the chicken, say 7lb, and write out the calculation you would do to find the cooking time: 30 + 20 × 7 = 170 minutes

**Stage 2**: Replace the weight you thought up with the letter W and then put T = in front of the formula

$T=30+20W$

Now, in algebra, people tend to write the term with the letter first and any 'constant' terms like the +30 in last, so

$T=20W+30$

is what you will find in GCSE textbooks.

**Challenge**: Google “GCSE Using formulae” and work through the BBC Bitesize GCSE Maths page. Try the questions.

## “Simplifying” algebraic expressions

The word 'simplify' is much overused in exam questions.

It means 'combine terms where you can to make the expression look shorter'.

You can't simplify $3x+5y$ but you *can* simplify $x+x+x+2y+3y$

Below are examples of the various kinds of simplification.

### *Collecting like terms*

**Example**: simplify $3x+4y-2x+7y$

**Stage 1**: You can break this expression into a set of terms; $+3x$ , $+4y$ , $-2x$ , $+7y$ .

The sign on each term is 'stuck' onto the term, so $-2x$ means 'subtract 2 lots of x'

**Stage 2**: You can rearrange the terms by letters so you have

$+3x \quad -2x \quad +4y \quad +7y$

**Stage 3**: You can add/subtract within the letter groups: $3x-2x=x$ and $4y+7y=11y$

**Stage 4**: Putting it all together the expression becomes $x+11y$

Powers of letters are different types of symbol when it comes to collecting terms

**Example**: Simplify $3x^2+5x-7x^2-3x+17$ as much as possible

**Stage 1**: Treat $x$ and $x^2$ as if they are different symbols, they cannot be added directly, and notice that 17 is the only pure number, so it will not combine with anything.

**Stage 2**: Collecting the like terms $3x^2-7x^2+5x-3x+17$

**Stage 3**: Add/subtract the like terms $3x^2-7x^2+5x-3x+17=-4x^2+2x+17$

**Stage 4**: Put into the conventional form (negative term after positive term)

$3x^2-7x^2+5x-3x+17=-4x^2+2x+17=2x-4x^2+17$

**Challenge**: Google “GCSE Collecting like terms” and work through the BBC Bitesize page including the questions.

## *Multiplying expressions*

**Example**: Multiply $4x^2\times-3y^4$

**Stage 1**: Work out the sign (plus times minus so minus)

**Stage 2**: Multiply the numbers (4 × –3 = –12)

**Stage 3**: Multiply the letters $x^2\times y^4=x^3y^4$

**Stage 4**: put it all together $4x^2\times-3y^4=-12x^3y^4$

You may have to multiply terms with the same letters

Recall that you can multiply powers of the same number by adding the powers.

This works for symbols as well: $x^2\times x^3=x^5$ and $x\times x=x^2$

**Example**: Multiply $x^3y^2\times xy^4$

**Stage 1**: Multiply the x letters $x^3\times x=x^4$

**Stage 2**: Multiply the y letters $y^2\times y^4=y^6$

**Stage 3**: put the expression together $x^3y^2\times xy^4=x^4y^6$

**Challenge**: Find an exercise about multiplying terms in a GCSE Foundation text book and try the questions. Check your anwers!

## *Multiplying out brackets*

Recall BIDMAS? So you could calculate $7\times(3+5)$ as 7 × 8 = 56

You can calculate this expression a different way $7\times(3+5)=7\times3+7\times5=21+35=56$

So if you have a term multiplying a bracket in an algebraic expression, you can multiply the term by each term within the bracket in turn.

**Example**: Multiply out $7(3\mathrm{x}+5)$

**Stage 1**: multiply each term within the bracket by the term outside the bracket

$7(3x+5)=7\times3x+7\times5=21x+56$

**Stage 2**: Simplify if possible by collecting terms

$21x+56$ can't be simplified

You can have several brackets together

**Example**: Multiply out this expression completely $4(3x+2)-2(x-5)$

**Stage 1**: multiply each term within the bracket by the term outside the bracket for each bracket in the expression

$4(3x+2)=4\times3x+4\times2=12x+8$

$-2(x-5)=-2\times x\,\mathrm{and}\,-2\times-5=-2x+10$

See that multiplying negative numbers is important in algebra

**Stage 2**: simplify by collecting terms

$12x+8-2x+10=12x-2x+8+10=10x+18$

**Challenge**: Google "GCSE multiplying out brackets" and try the BBC GCSE Bitesize page, both page 3 and page 4

## *Factorising*

Factorising is the reverse of multiplying out.

You find a factor of each term that you can put outside a bracket...

**Example**: Factorise $12x+15$

**Stage 1**: Both the +12x and the +15 terms have 3 as a factor

**Stage 2**: Put 3 outside the bracket and divide both the terms inside the bracket by 3

$12x+15=3(4x+5)$

You can have common factors with letters as well

**Example**: Factorise $10x^2+15x$

**Stage 1**: Common factors of both terms include 5 and also x

**Stage 2**: Put 5x outside the bracket and divide the terms inside the bracket by 5x

$10x^2+15x=5x(2x+3)$

**Check** by multiplying out $5x(2x+3)=5x\times2x+5x\times3=10x^2+15x$

### *Simplifying algebraic fractions*

Review the Number section about multiplying and dividing powers of the same number.

An algebraic fraction looks like this: $\frac{12x^3y^2}{4xy^2}$

You can simplify the numbers in the algebraic fraction easily enough, just do 12 ÷ 3 = 4

The letters can also be simplified using the rules of indices: $\frac{x^3y^2}{xy^2}=\frac{x\times x\times x\times y\times y}{x\times y\times y}=\frac{x^2}{1}$

Putting these together: $\frac{12x^3y^2}{4xy^2}=3x^2$

**Challenge**: Find exercises about collecting like terms, multiplying out brackets, factorising and simplifying algebraic fractions in a GCSE textbook and check the answers. Look for chapters with titles like 'basic algebra'.

# Equations

**Example**: Naomi thinks of a number, doubles it and adds three.

Her answer is 11.

What number did Naomi think of?

Most people will do 11 – 3 = 8 then do 8 ÷ 2 = 4.

You undid the add 3 by subtracting it and you then undid the doubling by dividing by 2

This word problem can be described in symbols as $2n-3=11$ and it is called an *equation*.

The equals sign is a symptom of an equation.

Naomi's equation is a two step equation because it has two operations.

You have an algebraic expression $2n-3$ that has been set equal to the number 11

There is only one value of $n$ that can make the equals sign true;

$2\times 4-3=11$

So $n=4$ is the *solution* to the equation.

Below is a series of examples of equations that get involve more and more steps.

**Challenge**: draw a function machine based on Naomi's puzzle 'double a number and add three'

## *One step equations*

A one step equation has just one operation.

These are all one step equations: $3x=15$ , $\frac{x}{8}=2$ , $x+3=9$ , $x-5=3$ , $\frac{2}{3}x=10$

You find the solutions by reversing the operations.

The detailed steps for each of these examples are below...

**Example**: Solve $3x=15$

**Stage 1**: 3 is multiplying the $x$ so divide by 3 on 'both sides' of the equal sign

$3x\div 3=x$ and $15\div 3=5$ so $x=5$

**Stage 2**: Check your answer by substituting $x=5$ into the equation $3\times 5=15$

So the solution $x=5$ must be right.

**Example**: Solve $\frac{x}{8}=2$

**Stage 1**: The 8 is dividing the $x$ so multiply both sides by 8

$\frac{x}{8}\times\frac{8}{1}=x$ and $2\times 8=16$ so $x=16$

**Stage 2**: Check: 16 ÷ 8 = 2 so solution correct

**Example**: Solve $x+3=9$

**Stage 1**: reverse the +3 by subtracting 3

$x+3-3=9-3$ so $x=6$

**Stage 2**: Check: 6 + 3 = 9 so solution $x=6$ must be correct

**Example**: Solve $x-5=3$

**Stage 1**: reverse the –5 by adding 5

$x-5+5=3+5$ so $x=3+5=8$

**Stage 2**: Check: $8-5=3$ so $x=8$ must be correct solution

**Example**: Solve $\frac{2}{3}x=10$

**Stage 1**: You can look at the $\frac{2}{3}$ as a fraction multiplying the $x$ , so then you divide both sides by $\frac{2}{3}$ . So $\frac{2}{3}x\div\frac{2}{3}=\frac{2}{3}\times\frac{3}{2}x=x$ and $10\div\frac{2}{3}=\frac{10}{1}\times\frac{3}{2}=\frac{30}{2}=15$ so $x=15$

**Stage 2**: Check: substitute 15 for x in the original equation $\frac{2}{3}\times\frac{15}{1}=10$

## *Two step equations*

A two step equation has two operations combined.

These are all two step equations: $2x+3=11$ , $4x-8=12$ , $1.2x+1.6=4$ , $3x+7=1$ and $3x+2=9$

You can solve these equations by treating them as a combination of two one step equations.

**Example**: Solve $2x+3=11$

**Stage 1**: Undo the +3 by subtracting 3 so $2x+3-3=11-3$ and $2x=8$

**Stage 2**: Undo the ×2 by dividing by 2, so $2x\div 2=8\div 2$ and $x=4$

**Stage 3**: Check the solution $x=4$ by substituting 4 back into the original equation

$2\times 4+3=8+3=11$ so $x=4$ is the correct solution.

**Example**: Solve $4x-8=12$

**Stage 1**: Undo the $-8$ by adding 8 to both sides $4x-8+8=12+8$ so $4x=20$

**Stage 2**: Undo the ×4 by dividing by 4 both sides $4x\div 4=20\div 4$ so $x=5$

**Stage 3**: Check the solution $x=5$ by substituting 5 back into the equation

$4\times 5-8=20-8=12$ so $x=5$ is the correct solution.

**Example**: Solve $1.2x+1.6=4$ (Yes, you can have decimals and fractions in equations)

**Stage 1**: Undo the +1.6 by subtracting 1.6 from both sides
$1.2x+1.6-1.6=4-1.6 \text{ so } 1.2x=2.4$

**Stage 2**: Undo the ×1.2 by dividing by 1.2 so $1.2x\div 1.2=x$ and $2.4\div 1.2=2$ so $x=2$

**Stage 3**: Check the solution by substituting $x=2$ into the equation

$1.2\times 2+1.6=2.4+1.6=4$ so the solution is correct.

Sometimes, you get solutions that are negative numbers

**Example**: Solve $3x+7=1$

Just see what the equation is saying: find three times some number, then *add 7*, and the answer just makes it to 1. The number you started with just has to be less than zero.

**Stage 1**: Undo the +7 by subtracting 7 from both sides $3x+7-7=1-7 \text{ so } 3x=-6$

**Stage 2**: Undo the ×3 by dividing both sides by 3 $3x\div 3=-6\div 3 \text{ so } x=-2$

**Stage 3** : Check the solution by substituting back into the equation

$3\times -2+7=-6+7=+1$ so $x=-2$ is confirmed as the correct solution

And sometimes you get equations where the answer is a fraction

**Example**: Solve $3x+2=9$

Stage 1: Undo the +2 by subtracting 2 from both sides $3x+2-2=9-2 \text{ so } 3x=7$

**Stage 2**: Undo the ×3 by dividing by 3 both sides $3x \div 3=7 \div 3 \text{ so } x=2\frac{1}{3}$

**Stage 3**: Substitute $x=2\frac{1}{3}$ into the original equation

$3\times 2\frac{1}{3}+2=\frac{3}{1}\times\frac{7}{3}+2=7+2=9$ and the solution is correct.

## *Equations with letters on both sides*

The next most complicated kind of equation has the unknown on both sides. Some textbooks refer to these as 'double sided equations'.

The best approach is to simplify the equation into one with letters just on one side by removing the smallest term.

**Example**: Solve $2x+5=x+4$

**Stage 1**: Subtract $x$ from both sides of $2x+5=x+4$

$2x+5-x=x+4-x$ gives $x+5=4$ .

We picked $x$ as the term to subtract because it is smaller than $2x$

See how you end up with a one step equation to solve?

**Stage 2**: Solve $x+5=4$ by subtracting 5 from both sides

$x+5-5=4-5$ gives $x=-1$

**Stage 3**: Check the solution by substituting $x=-1$ back into the original equation

$2\times -1+5=-1+4$ gives $-2+5=-3 \text{ so } -3=-3$

We know that $x=-1$ is the correct solution to the equation because one side ends up equal to the other side when you substitute the value in.

**Example**: Solve $2x+10=7x-5$

**Stage 1**: Subtract $2x$ from both sides because $2x$ is the smallest $x$ term.

$2x+10-2x=7x-5-2x$ Gives $10=5x-5$

**Stage 2**: make $10=5x-5$ look like a 2 step equation by turning it round, $5x-5=10$

**Stage 3**: Add 5 to both sides of $5x-5=10$ gives $5x-5+5=10+5$ so $5x=15$

**Stage 4**: Divide both sides by 5 gives $5x\div 5=15\div 5$ so $x=3$

**Stage 5**: Check the solution by substituting $x=3$ into $2x+10=7x-5$

$2\times 3+10=7\times 3-5$ so $6+10=21-5$ so $16=16$ and the solution $x=3$ is correct

## *Equations with brackets*

You can have expressions with brackets in equations.

The easiest way to deal with these is to multiply out the brackets.

You might be able to use division to solve the equation more quickly.

**Example**: Solve $3(2x+1)+5=20$

**Stage 1**: Multiply out the bracket and collect terms on the right hand side
$3(2x+1)+5=6x+3+5=6x+8$

**Stage 2**: The left hand side is still 20, so the equation is now $6x+8=20$ .

This is a two step equation and you can solve that as usual

**Stage 3**: Subtracting 8 from both sides $6x+8-8=20-8$ gives $6x=12$

**Stage 4**: Dividing both sides by 6 gives $6x\div 6=12\div 6$ so $x=2$

**Stage 5**: As always, you can check the result by substituting $x=2$ into the original equation. $3(2\times 2+1)+5=3\times 5+5=15+5=20$

**Example**: Solve $2(4x+1)-3(2x-5)=27$

**Stage 1**: Multiply out both brackets and collect the terms together

$2(4x+1)-3(2x-5)=2\times 4x+2\times 1-3\times 2x+3\times 5=8x-6x+2+15=2x+15$

(See how that one line tested most of the 'simplifying' section of Algebra)

**Stage 2**: Solve the simplified equation $2x+15=27$

**Stage 3**: Subtracting 15 from both sides $2x+15-15=27-15$ gives $2x=12$

**Stage 4**: Dividing both sides by 2 so $2x\div 2=12\div 2$ gives $x=6$ as the solution

**Stage 5**: Substitute $x=6$ into the original equation to check

$2(4\times 6+1)-3(2\times 6-5)=48-21=27$ so the solution is correct

Example: Solve $3(2x-7)=15$

Both sides of this equation can be divided by 3, so we can take a short cut

**Stage 1**: Divide both sides by 3 to simplify the equation

$3(2x-7)\div 3=15\div 3$ gives $2x-7=5$

**Stage 2**: Now solve the two step equation by adding 7 to both sides

$2x-7+7=5+7$ gives $2x=12$

**Stage 3**: Then divide both sides by 2

$2x\div 2=12\div 2$ gives $x=6$ as the solution

**Stage 4**: Check by substituting $x=6$ into the original equation

$3(2\times 6-7)=3(12-7)=3\times 5=15$ so $x=6$ is the correct solution

**Mega challenge**: Find the equations chapter in a GCSE Foundation Maths textbook and work through as many kinds of equation as you can. This will pay dividends!

# Inequalities

## *Greater than and less than signs*

**Example**: “3 < 5” reads “Three is less than five”.

**Example**: “12 > 7” reads “Twelve is more than seven”

**Example**: “x ≥ 3” reads “X can be greater than or equal to 3” which means that X can be any number from 3 upwards, like 3.05 or 10 000.

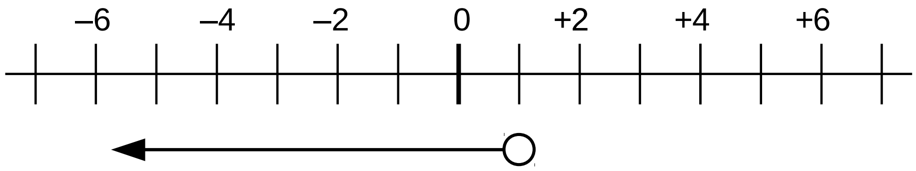
**Example**: “Y ≤ 2” reads “Y can be less than or equal to 2” which means that Y can be any number from 2 downwards, like 2 or 1.99 or 0 or –8

The 'point' of an inequality sign points to the smaller, the 'mouth' points to the larger

## *Number line notation*

You can use a number line to show the range of an inequality.

**Example**: Draw the inequality  X < 1 on the number line below

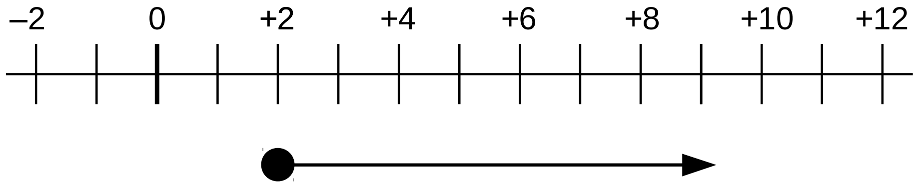



The open circle is drawn opposite 1 on the scale.

An open circle means strictly less than or strictly equal to

The arrow is pointing in the negative direction, so in this case the open circle means 'less than'.

**Example**: Draw the inequality Y ≥ 2 on a number line

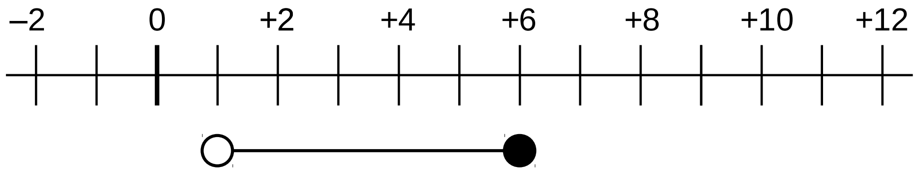



The closed circle is drawn opposite to the +2 point on the number line

A closed circle says that the inequality can be equal to as well as greater or less than

As the arrow is pointing in the positive direction, in this case the filled circle means “greater than or equal to”

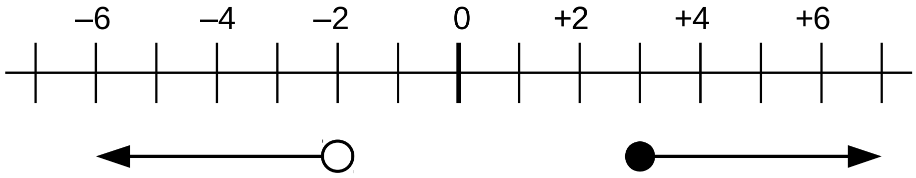
**Example**: Show the inequality 1 < X ≤ 6 on a number line




The '1 less than X' part is represented by the open circle by the +1 on the number line

The 'X less than or equal to' part is represented by the filled circle by +6

**Example**: Show the inequality Y < –2 and Y ≥ 3 on a number line

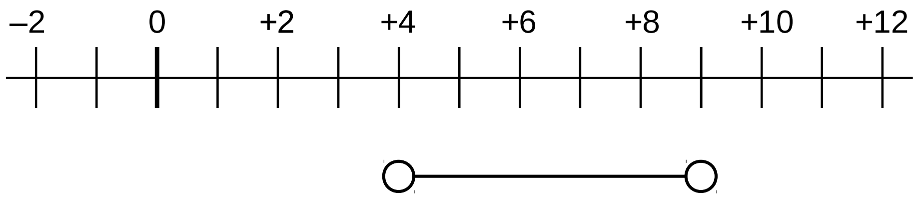



The Y < –2 part is an open circle opposite –2 with an arrow pointing negative

The Y ≥ 3 part is a filled circle opposite +3 with an arrow pointing positive

The diagram shows you that Y can be any number except in the range –2 to just under 3.

**Example**: Write the inequality represented by this diagram




Use the letter X for the unknown

The bottom open circle is opposite 4, so that part means 4 < X

The upper open circle is opposite 9, so that part means X < 9

The line between the open circles means that X can vary between just over 4 and just under 9

So the inequality is written 4 < X < 9

## *Integers that fit a given inequality*

An integer is a whole number that can be negative, zero or positive.

Integers include –20, –7, 0, 5, 57, 1000

Numbers that are *not* integers include –2.5, 7.3, $\frac{3}{4}$

**Example**: What is the smallest integer that satisfies the inequality –5 < X ≤ 10?

Smallest, so we look at the lower bound, –5

The inequality there is <, so X can't take on the value –5

The next possible integer is –4

**Example**: List the integers N that are consistent with –12 ≤ N < –7

–12 is included because of the 'less than or equal' sign, and –7 isn't included because of the < sign.

The possible integers are –12, –11, –10, –9, –8

## *Solving inequalities*

**Example**: Solve the inequality $2x+3<11$

Just treat the < as if it was an = and work through the same steps you would take to solve the equation

**Stage 1**: Subtract 3 from both sides $2x+3-3<11-3$ gives $2x<8$

**Stage 2**: Divide both sides by 2 so $2x\div 2<8\div 2$ gives $x<4$

**Stage 3**: Substitute $x=4$ into the left hand side of the inequality $2\times 4+3=11$ so we know the boundary is OK.

Substitute a value less than 4 into the left hand side of the inequality, say $x=1$

$2\times 1+3=5$ and 5 is certainly less than 11, so our solution to the inequality is correct.

**Challenge**: Google “GCSE Inequalities” and work through the BBC Bitesize page that is listed in the search results. Inequalities and graphs are covered later in this booklet.

# Linear Sequences

A sequence of numbers is a list of numbers that follow some sort of pattern.

**Example**: 4, 7, 10, 13...

The list above shows the first four **terms** of a sequence..

Each term of the sequence is three larger than the last term.

You are adding 3 each time. Because you always add 3, this is a linear sequence.

The first term of the sequence is 4, and the 3rd term of the sequence is 10

The next three terms of this sequence are 16, 19, 22

**Example**: 12, 10, 8, 6...

The list above shows the first four terms of another sequence.

The **term to term** rule is “the next term is found by subtracting 2 from the current term”

The next four terms of this sequence are 4, 2, 0, −2

After the 7th term, the terms are negative.

## *The nth term formula*

Suppose you had the sequence 7, 11, 15, 19...

What is the value of the 2000th term?

The best way to work out the value is to write a formula for the value of a term in terms of the number of the term, and then substitute the value n=2000 into the formula.

Textbooks often use the letter n to represent the number of the term, so the formula has become known as the nth term formula.

**Example**: find a formula for the value of the $n^{th}$ term of the sequence 7, 11, 15, 19 and use your formula to find the value of the $2000^{th}$ term of the sequence

**Stage 1**: Find the difference between successive terms.

In this case, the difference is $11 - 7 = 4$

**Stage 2**: Make a list of the multiples of the difference and compare those to the original sequence.

| n | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Term | 7 | 11 | 15 | 19 | 23 |
| 4n | 4 | 8 | 12 | 16 | 20 |

The formula for the multiples is 4n, just the difference times the term number

**Stage 3**: Compare the multiples with the values of the terms and adjust the formula

The values in the 4n row are too small by 3 in each case, so we change the formula for the $n^{th}$ term to 4n+3...

| n | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Term | 7 | 11 | 15 | 19 | 23 |
| 4n | 4 | 8 | 12 | 16 | 20 |
| 4n+3 | 7 | 11 | 15 | 19 | 23 |

**Stage 4**: substitute into the formula to find the value of the $2000^{th}$ term

Term value = $4n + 3$, and $n=2000$

Term value = $4 \times 2000 + 3 = 8003$

Sometimes, the multiples are more than the terms in the original sequence

**Example**: Find the formula for the value of the $n^{th}$ term of the sequence 4, 11, 18, 25...

**Stage 1**: The difference between terms is 7

**Stage 2**: The table of multiples (7n) looks like this

| n | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Term | 4 | 11 | 18 | 25 | 32 |
| 7n | 7 | 14 | 21 | 28 | 35 |

**Stage 3**: Comparing the values of the 7n row with the original terms shows that we have to subtract 3 from the 7n row to match the original sequence.

The formula for the nth term is $7n - 3$

**Stage 4**: Check the formula by substituting in $n = 4$

$7 \times 4 - 3 = 28 - 3 = 25$ which matches the $4^{th}$ term of the original sequence

### *Puzzle questions*

Examiners seem to like these.

You have to think a bit outside the formulas.

Many of the puzzle questions involve the formula for the nth term...

**Example**: Can 240 be a term in the sequence 8, 14, 20, 26?

**Stage 1**: find the formula for the value of the $n^{th}$ term for the sequence

The difference between the terms is 6

The multiples (6n) are 6, 12, 18, 24

We have to add 2 to each multiple to get the value of the corresponding term in the sequence

So the formula for the value of the $n^{th}$ term is 6n + 2

**Stage 2**: Set the formula for the nth term equal to 240 and solve the equation

If 240 is a term in the sequence, then 6n + 2 = 240 has a solution that is a whole number.

6n + 2 =240 so 6n = 238 and n = 238/6 = 39.666...

So 240 can't be a term in the sequence.

**Challenge**: Google “GCSE Sequences” and try both pages of the BBC Bitesize page. Not all sequences are linear, you need to be able to recognise common non-linear sequences like the square numbers.

## Trial and improvement

One way of solving hard equations is by guessing, then comparing what happens when you substitute in your guess with the answer you are trying to reach.

### *Guessing and comparing*

**Example**: Solve $x^3=10$

**Stage 1**: make a guess, say x = 2

**Stage 2** Substitute your guess into the left hand side $2^3=2\times2\times2=8$ (the 'trial')

**Stage 3**: Compare 8 with 10: your guess is too low, so pick a larger value, say x =3 (the 'improvement')

**Stage 4**: Substitute the larger guess into the LHS $3^3=3\times3\times3=27$

**Stage 5**: Compare, 3 is too large. The real answer must be closer to 2 than to 3 because 8 is much closer to 10 than 27 is

**Stage 6**: Make a new guess, say x= 2.3

**Stage 7**: Substitute this new guess into the LHS $2.3^3=2.3\times2.3\times2.3=12.167$

**Stage 8**: Compare, x=2.3 is too large so try x= 2.1

**Stage 9**: Substitute new guess into LHS $2.1^3=2.1\times2.1\times2.1=9.621$

**Stage 10**: Compare, x = 2.1 is too small but close, so try x = 2.15

**Stage 11**: Substitute new guess into LHS $2.15^3 = 2.15 \times 2.15 \times 2.15 = 9.938375$

**Stage 12**: very close, so $x^3 = 10$ has a solution close to x = 2.15

As you can see, you are repeating three steps; making a guess, calculating the value of $x^3$ and comparing the value of $x^3$ with 10. Based on the comparison, you make a new guess.

You can use this procedure with more complex equations...

## *More complex equations, using a table to collect trials*

**Example**: Solve $x^3 - 5x = 26$ accurate to two decimal places using the trial and improvement method

**Stage 1**: Make an initial guess, say x = 3 (I chose that because $3^3 = 27$ and 5×3=15)

**Stage 2**: Substitute x =3 into the formula $x^3 - 5x$ gives $3^3 - 5 \times 3 = 27 - 15 = 12$ When you compare 12 with 26, the result is too small by 14

**Stage 3**: Try x = 4. $4^3 - 5 \times 4 = 64 - 20 = 44$ and 44 is too large by 18

**Stage 4**: We know the answer is between 3 and 4, and might be closer to 3. Lets set up a table to organise the trials and improvements with this more complex formula. Each row of the table is a new guess, the trial of that guess by substituting and a comparison.

| **(New) Guess** | **Substitute into** $x^3 - 5x$ | **Compare with 26** |
|---|---|---|
| 3.5 (midpoint of 3 and 4) | $3.5^3 - 5 \times 3.5 = 42.875 - 17.5 = 25.375$ | Too small but close |
| 3.6 (a bit larger than 3.5) | $3.6^3 - 5 \times 3.6 = 46.656 - 18 = 28.656$ | Too large and not so close |
| 3.55 (midpoint of 3.5 and 3.6) | $3.55^3 - 5 \times 3.55 = 26.988875$ | Still too large |
| 3.52 (closer to 3.5) | $3.52^3 - 5 \times 3.52 = 26.014208$ | Pretty Close! |

So the solution to $x^3 - 5x = 26$ is $x = 3.52$ (don't put the 26.01... down as the answer, that was just the figure you were comparing with to refine your guesses).

You need to make sure that you can correctly substitute into formulas using BIDMAS rules.

**Challenge**: Google “GCSE Trial and improvement” and work through the BBC Bitesize GCSE Maths page.

# Graphs

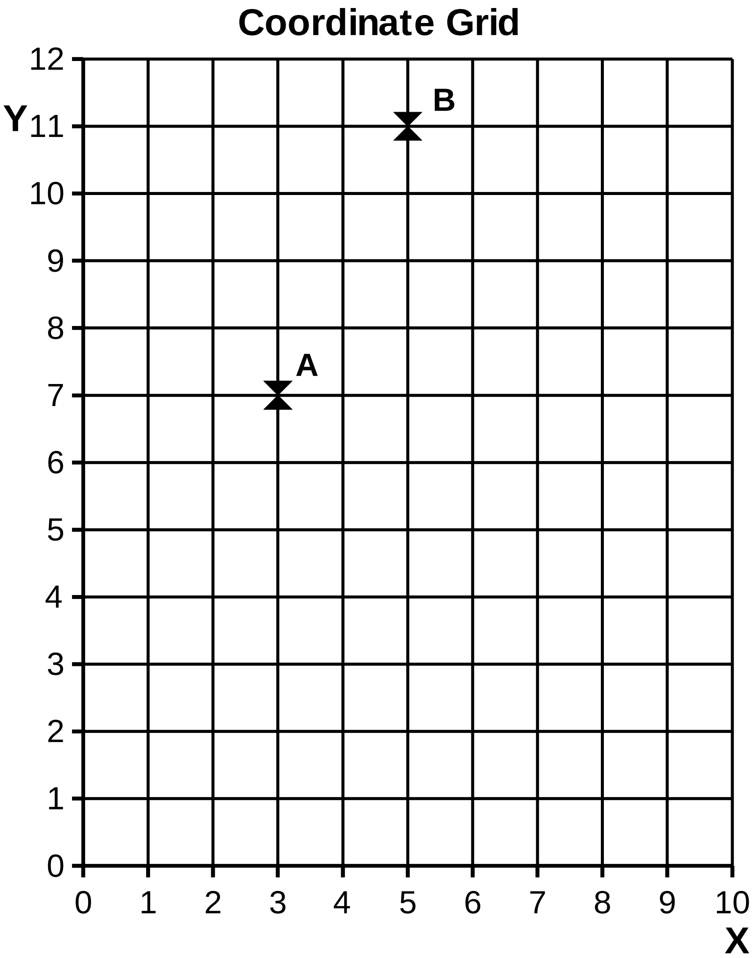
Mathematical graphs let you turn formulas into pictures.

'Everyday graphs' are used to convey information about distances and prices quickly.

## Coordinates

### *Plotting points and reading coordinates*

Look at the coordinate grid below




The <u>horizontal axis</u> is referred to as the X axis and <u>vertical axis</u> is referred to as the Y axis.

The **origin** is in the bottom left corner and has **coordinates** (0,0)

Point A has **coordinates** (3,7) and point B has **coordinates** (5,11)

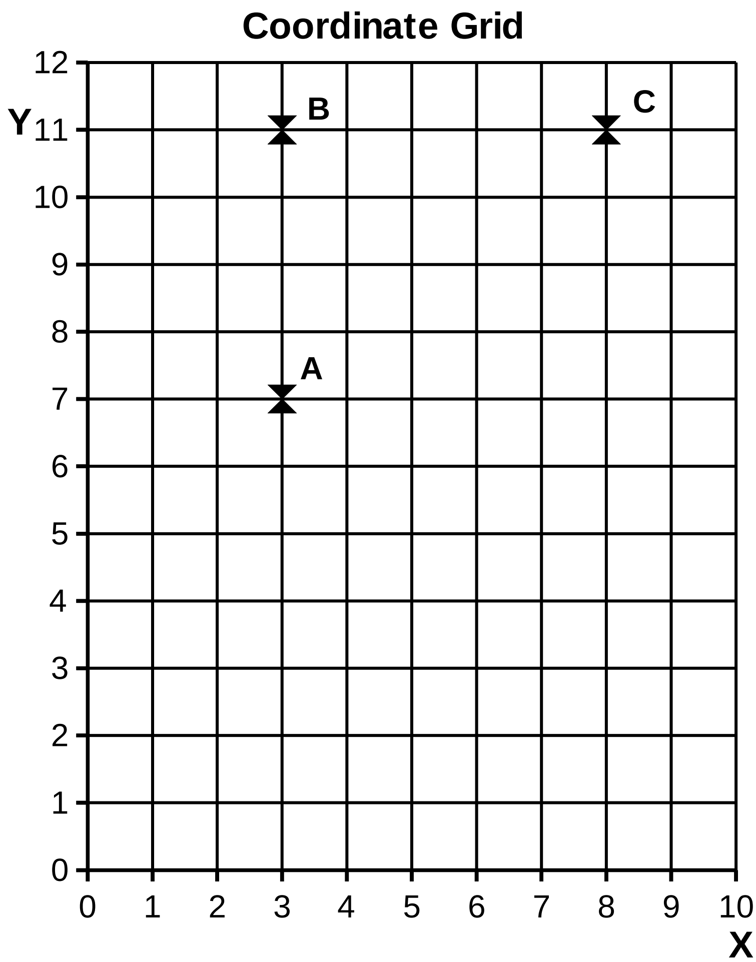
You always put the coordinates of a point in brackets separated by a comma, and you always write the **X coordinate** on the left and the **Y coordinate** on the right

To plot points from coordinates, you "**walk along the corridor and go up the stairs**"

**Challenge**: Draw a shape on squared paper and draw a set of axes. Write down the coordinates of your shape on another piece of paper. Swap just your coordinates with someone else. Draw their shape. Then compare your shapes!

## *Completing shapes on coordinate grids*

**Example**: Points A, B, C, D form the vertices of a rectangle. Points A, B, C are drawn on the grid below. Write down the coordinates of point D.




The vertices of a shape are its corners. One corner is called a vertex.

The fourth point has to have X coordinate 8 so it lines up with C and Y coordinate 7 so it lines up with A.

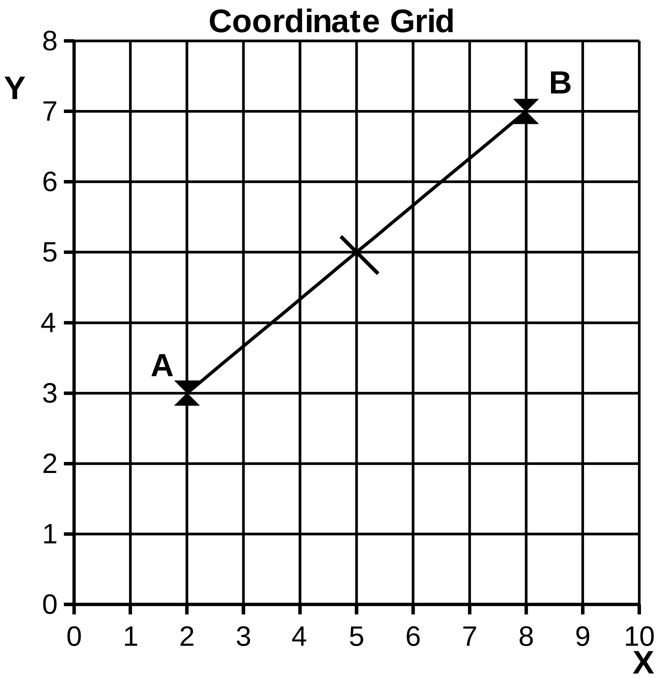
The coordinates of point D must be (8, 7)

Examiners are fond of mixing coordinates and shape and space, so this question might continue by asking the area of the rectangle A, B, C, D and even the length of the diagonal A, C.

See the Shape and Space section for details.

## *Midpoint of two points*

**Example**: Draw the midpoint of the points A and B on the grid below. Write down the coordinates of the midpoint.



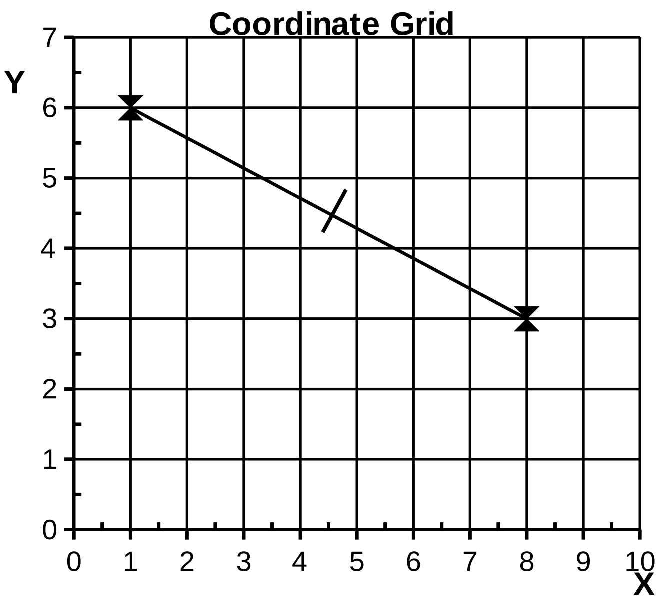

The 'midpoint' is just the point in the middle.

The midpoint of the line AB is 3 squares along and 2 squares up from point A and has coordinates (5,5), see the mark in the middle of the line.

You can have fractional coordinates.

**Example**: Point C has coordinates (1,6) and point D has coordinates (8,3)

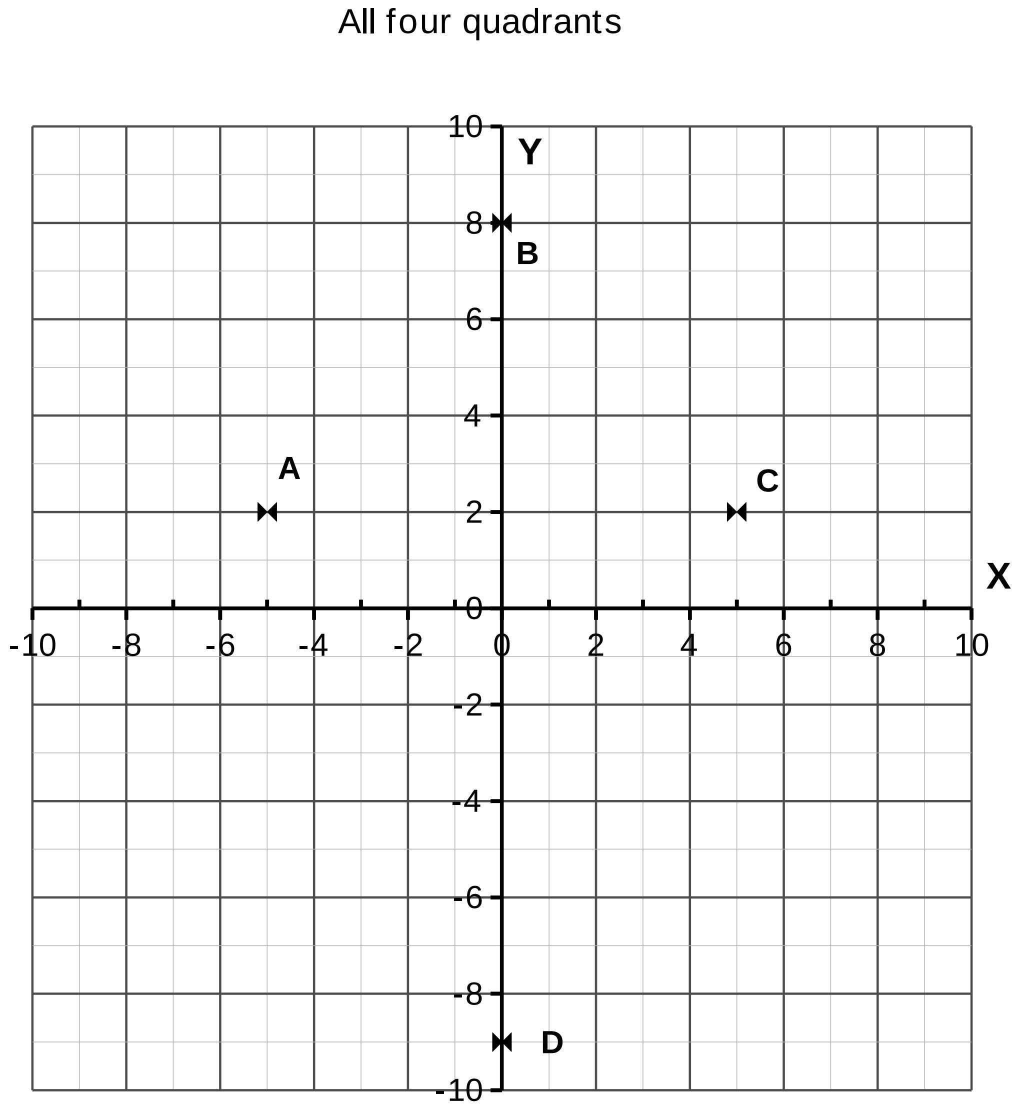
Write down the coordinates of the midpoint of CD




The midpoint is 3½ squares along and 1½ squares down from point C, so the coordinates of the midpoint are (4½,4½).

## *Negative and positive axes*

**Example**: Read off the coordinates of A, B, C, D on the grid below




See the X axis has negative numbers on the left, then zero then positive numbers on the right of the origin, just like a number line.

See the Y axis is like a thermometer scale, with positive coordinates above zero and negative below zero

**Finding coordinates**: Read the X coordinate first and then the Y coordinate

A has X coordinate –5 and Y coordinate +2 which are written as (–5, +2)

B has X coordinate 0 because it is on the X axis itself, so the coordinates are (0, 8)

C has coordinates (5, 2) and D has coordinates (0, –9)

Join ABCD together with lines, they make a kite with the Y axis as mirror line (see Shape)

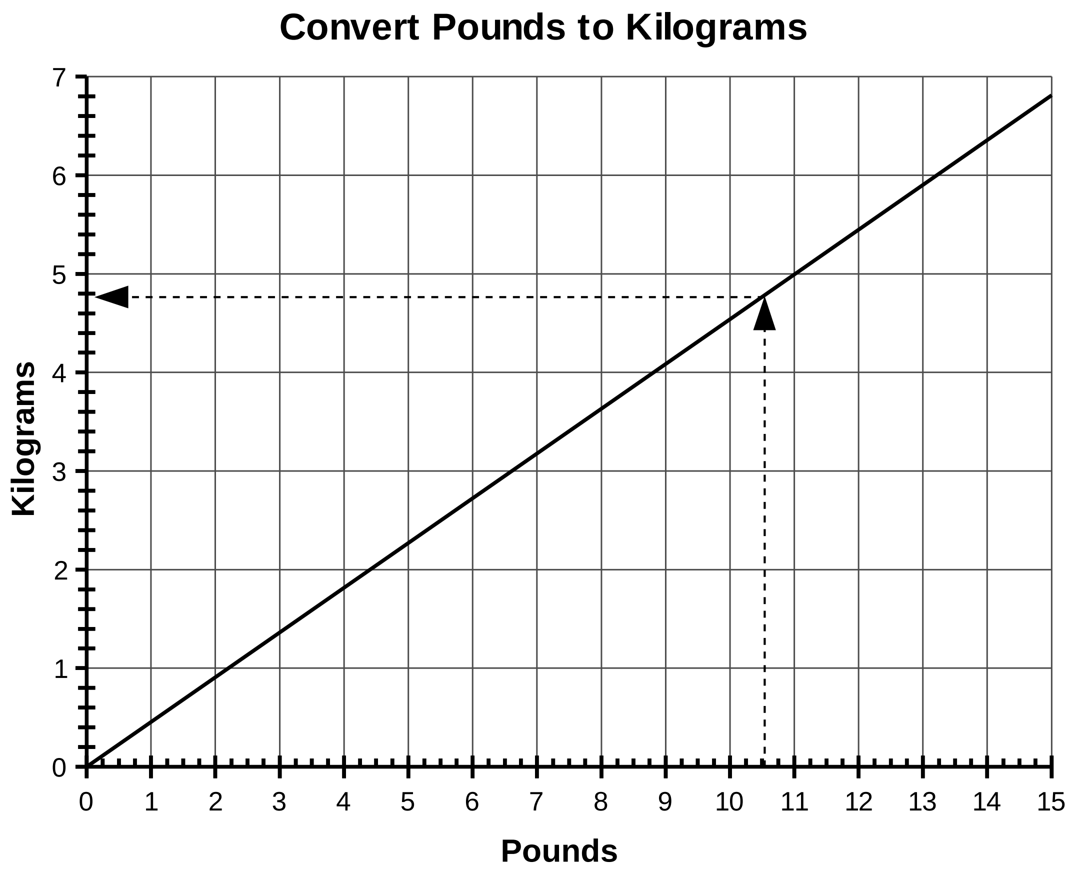
# Everyday graphs

Graphs are used in catalogues, price plans and in health education information. You can read off numbers from the graphs.

## *Conversion graphs*

Conversion graphs lets you change between metric and imperial units by drawing lines and reading off a scale.

**Example**: Use the graph below to convert 10½ Pounds to Kilograms




**Stage 1**: Read the axes carefully and check how the scale works.

The horizontal axis shows the pounds weight, and each pound is broken into quarters (4 ounces).

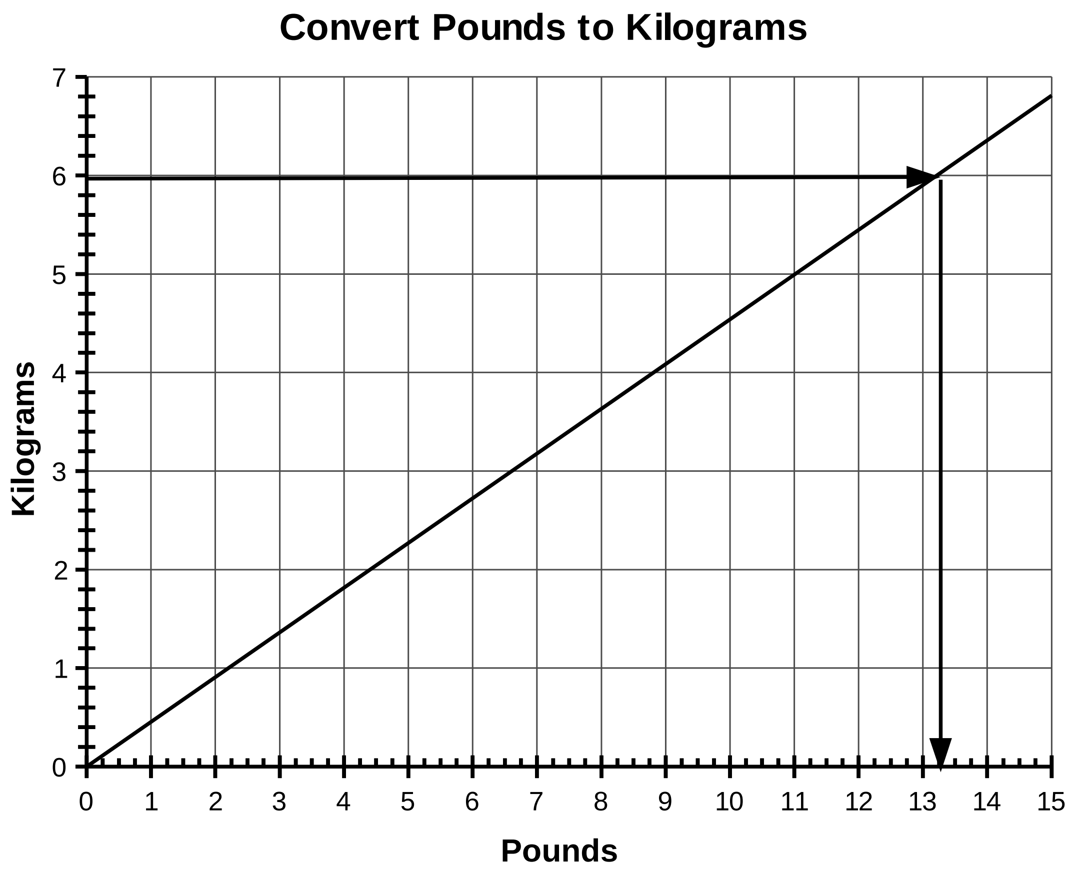
**Stage 2**: Locate the weight in pounds and draw a line up from the horizontal scale until the line reaches the sloping line on the graph

**Stage 3**: Draw a horizontal line along from where your vertical line crosses the sloping line to the vertical axis

**Stage 4**: Read the value of Kg from the vertical axis, notice that each kilogram is broken into 5 divisions, each worth 0.2Kg

By my eyesight, the vertical scale value corresponding to 10½ pounds is 5.8Kg

**Example**: Use the graph to estimate the number of pounds in 24Kg




**Stage 1**: The number of kilograms is larger than the scale, so we pick a factor of 24Kg, say 6Kg. We've divided the original quantity by 4.

**Stage 2**: Locate 6Kg on the vertical scale and draw the line along to the sloping line

**Stage 3**: Draw a line down from where the line along meets the sloping line

**Stage 4**: Read off the value in pounds as before, I read it as 13¼ pounds

**Stage 5**: Multiply the value you read off in pounds by 4, so 4 × 13¼ = 53lb

**Challenge**: find a GCSE Maths textbook and find as many different conversion charts as you can. Practice using the charts to convert to and from units. Examples will probably include

- Temperature (Fahrenheit to Celsius)
- Miles and Kilometres
- Litres and pints
- Centimetres to feet and inches

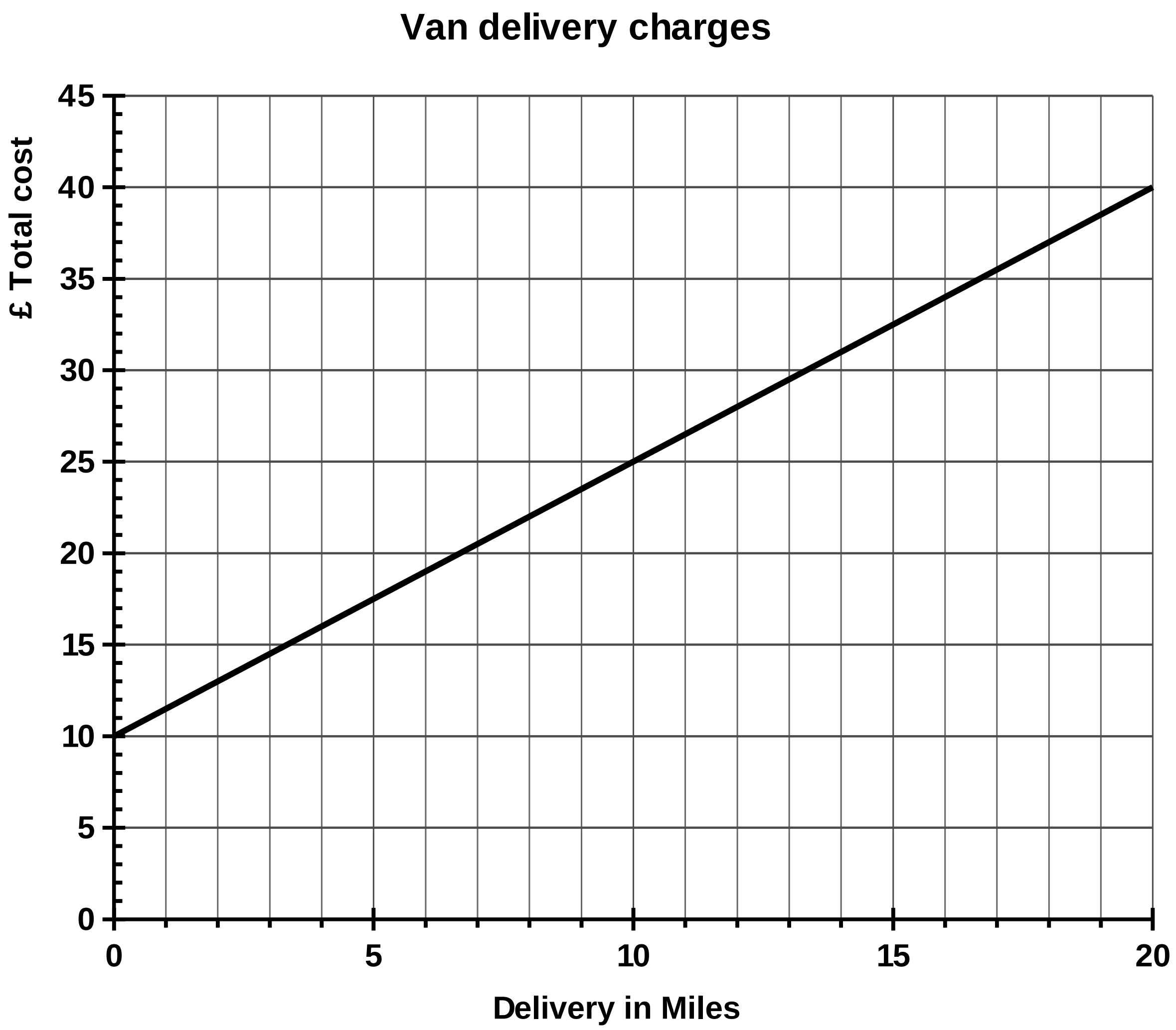
See the Metric units section in Shape for more about units.

## *Fixed and variable cost graphs*

A mobile phone contract will cost so much per month with a certain number of free minutes, and then the customer has to pay for each 'extra' minute used in that month.

The cost per month is called the 'fixed cost' and the cost for each minute over the free allowance is called the 'variable cost'.

Below is a graph showing the total cost of a local van delivery.




**Example**: How much does a journey of 13 miles cost?

**Stage 1**: Find 13 miles on the horizontal axis

**Stage 2**: Draw a line up and across to the vertical axis and read off the cost

I make the cost £29.50

**Example**: How far can you go for £25?

**Stage 1**: Find £25 on the total cost axis

**Stage 2**: Draw a line across and down and read off the distance

I make the distance 10 miles

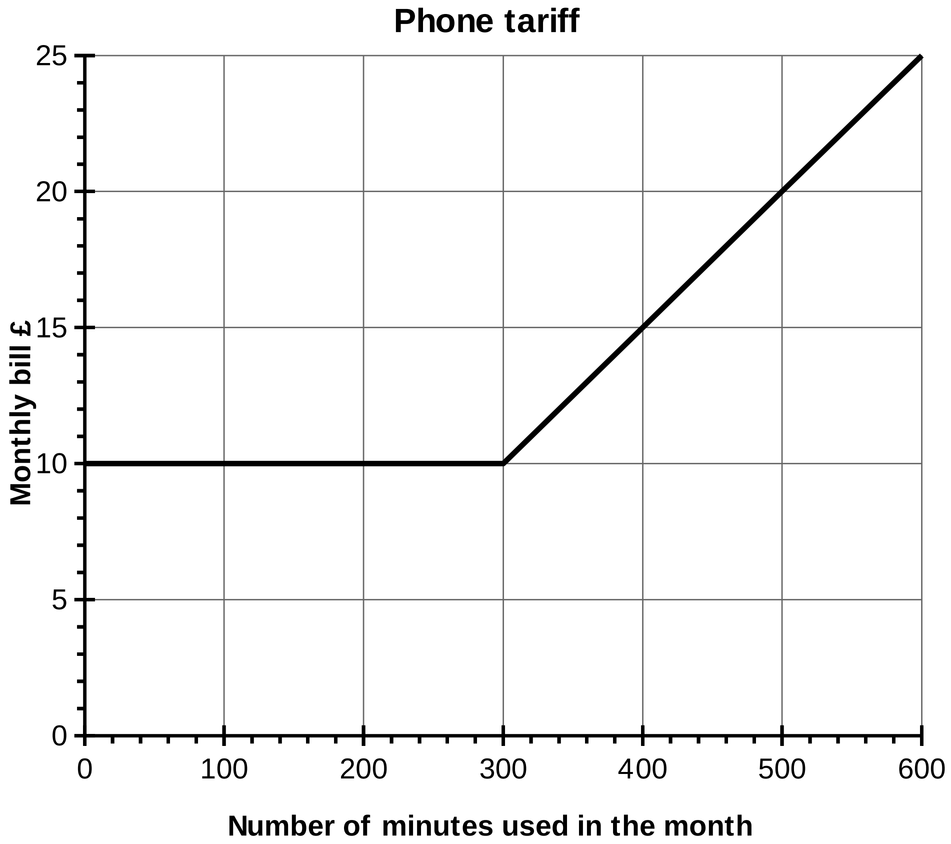
## *Tariff graphs*

A <u>tariff</u> is the name for a set of rules for charging for something. It could be electricity, gas or mobile phone costs.

Suppose a mobile phone company charged £10 per month, and gave 300 minutes of phone time free each month.

Suppose they charged 5p per minute for phone minutes above the 300 free minutes.

A graph of the monthly cost against total number of minutes used would look like this;




**Example**: Freda uses 440 minutes one month. What is her bill?

**Stage 1**: go along to 440 minutes on the horizontal axis

**Stage 2**: draw a line up to the cost graph, then across to the vertical axis and read off the cost.

I make the cost £17

**Example**: Mkhosi wants to pay a maximum of £12 per month for his mobile phone service. How many minutes can he use?

**Stage 1**: Find £12 on the vertical axis

**Stage 2**: Draw a line along until it meets the cost graph, then down to the horizontal axis and read off the number of minutes.

I estimate the number of minutes to be 340 minutes.

**Challenge**: Find some examples of 'tariff graphs' in a Maths textbook and see if you can answer the questions.

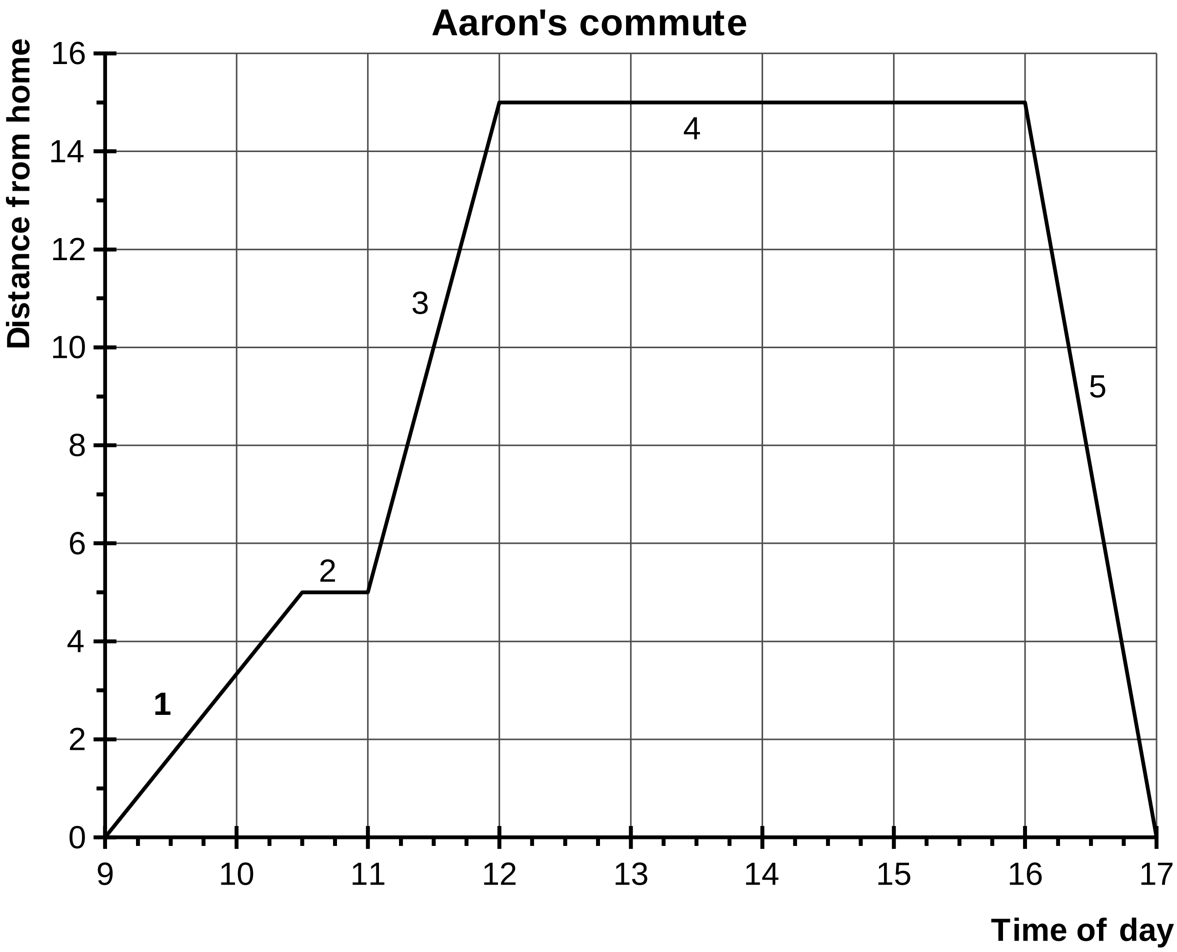
## *Distance time graphs*

You can use a graph to tell the story of a journey.

The vertical axis of a distance-time graph shows the distance away from the starting point of the journey.

The horizontal axis shows 'clock time'

Below is a distance time graph that shows Aaron's journey from Birmingham to Walsall each day.




The graph (and journey) can be broken up into 5 stages (numbered 1 to 5 on the graph).

1. The train is running very slowly.
2. Aaron stops at a station to buy a sandwich. The horizontal line shows that he has not travelled any further. The stop takes half an hour.
3. The next train is much quicker as shown by the steeper line.
4. Aaron stays in Walsall for four hours from 12 noon to 1600 (24 hour clock)
5. Aaron catches the express train home. No stopping this time!

Recall Speed, Distance and Time in Algebra. Aaron's speed in section 3 is 10 miles (15 – 5) in one hour, so 10 mph.

**Challenge**: find some distance-time graphs in a GCSE textbook. Answer the questions.

# Mathematical graphs: 'special lines'

Mathematical graphs provide you with a 'picture' of a formula like $y=3x+2$ or $y=x^2-3x+5$ .

You need to recall the 'coordinates' work to make sense of these graphs.

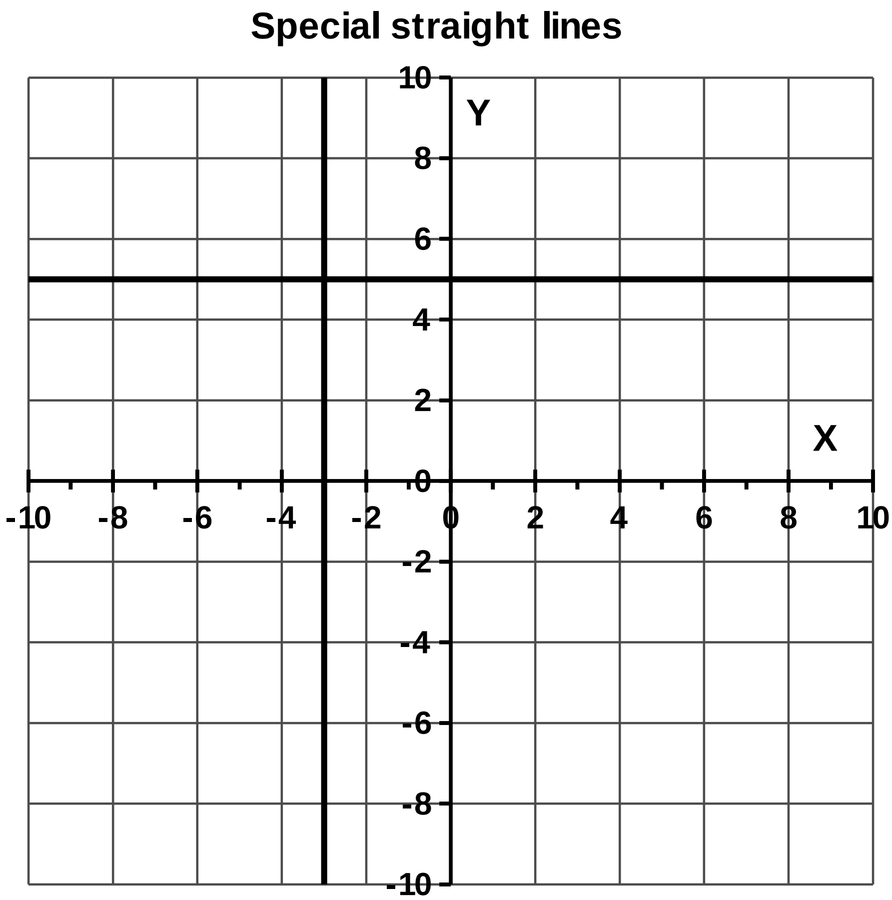
Let's start with how you write a formula for a horizontal straight line and a vertical straight line on a set of graph axes.

**Example**: Draw the line Y = 5 and the line X = –3 on a coordinate grid

**Stage 1**: The line Y = 5 means that any point on the line has to have Y coordinate 5, but can have any X coordinate. The result is a horizontal line going through 5 on the Y axis

**Stage 2**: The line X = –3 means that any point on the line has to have X coordinate –3 but can have any Y coordinate. The result is a vertical line going through –3 on the X axis

See below for the completed coordinate grid




The Y axis is the line Y = 0 and the X axis is the line X = 0

You can use mathematical formulas or rules to describe straight lines that slope on graphs, see the next section.

# Straight line graphs

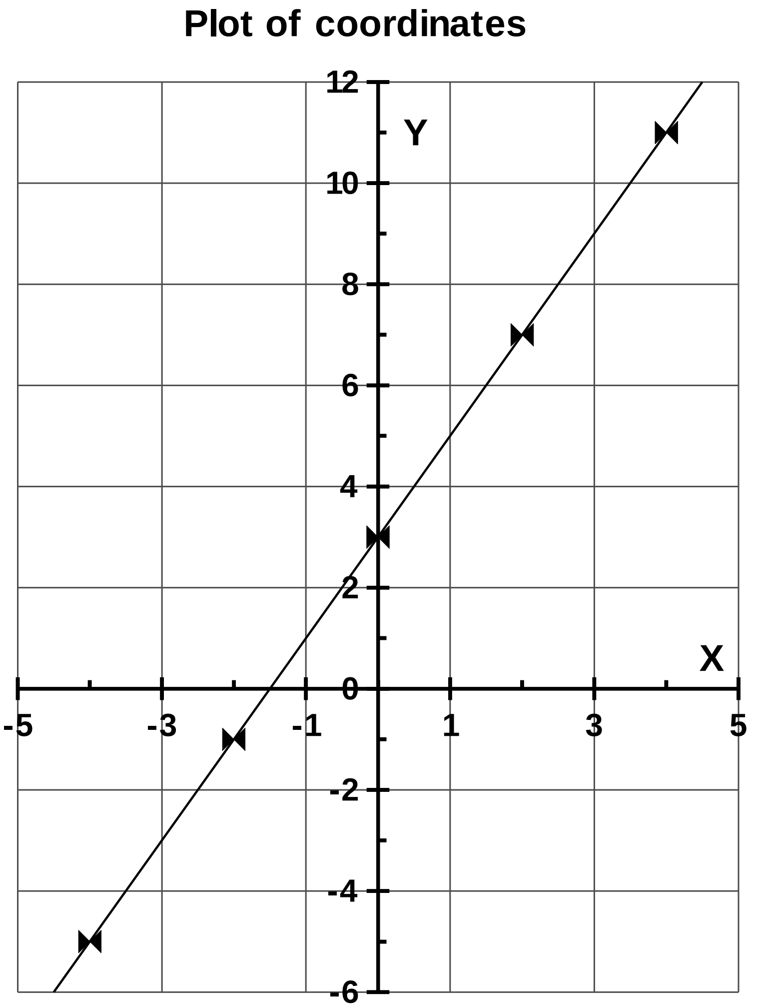
**Example**: Plot the points in the table below on a coordinate grid and draw a line through the points.

| X | –4 | –2 | 0 | 2 | 4 |
|---|---|---|---|---|---|
| Y | –5 | –1 | 3 | 7 | 11 |

**Stage 1**: Plot each column in the table as a point, so the coordinates of the first point are (–4, –5), and then (–2, –1) and then (0, 3) and then (2, 7) and finally (4, 11)

**Stage 2**: Draw a straight line through the points.

If you can't draw a single straight line through the points, check your plotting!

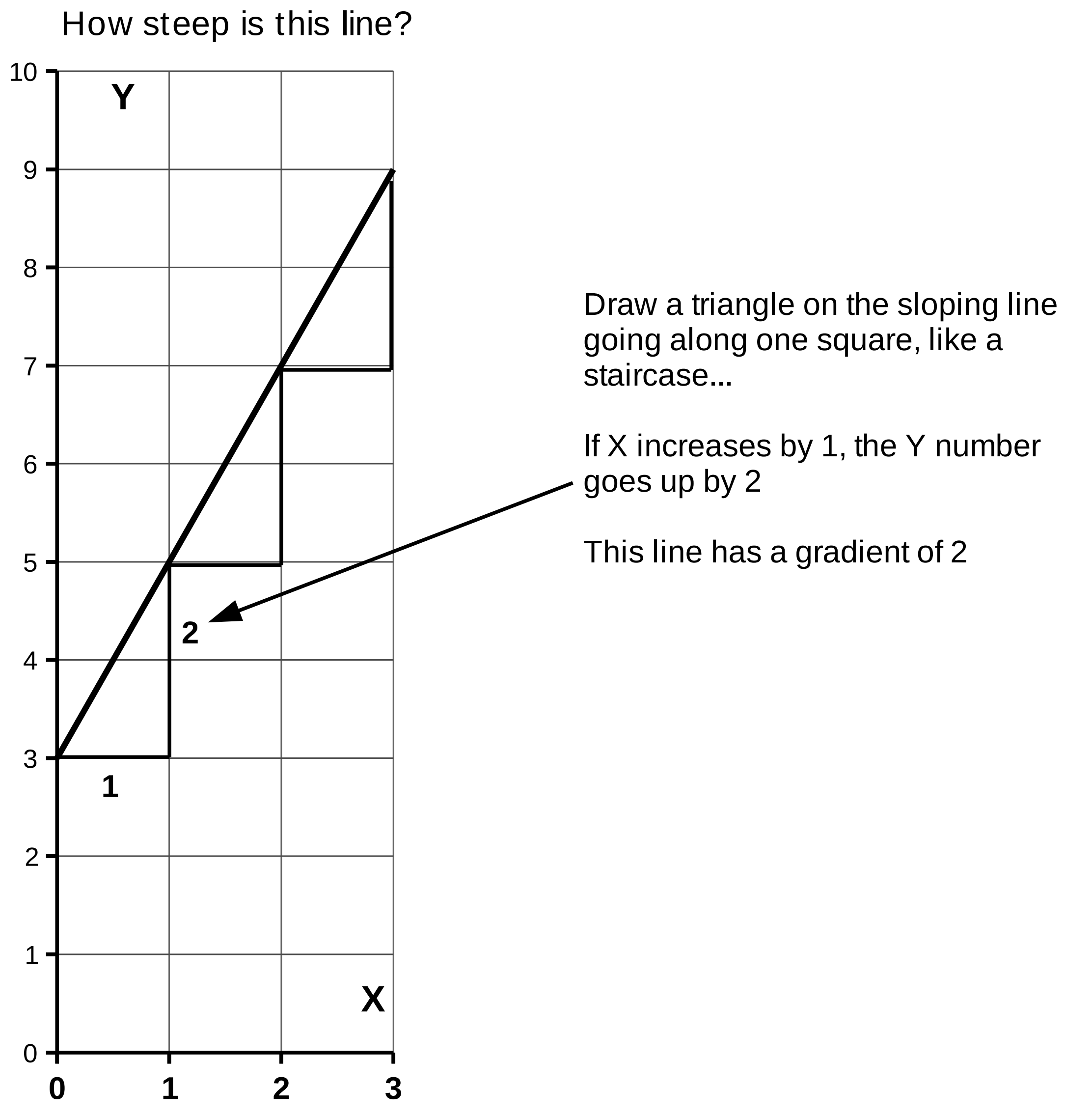


## *Gradient*

The gradient of a straight line is a number that measures how steep the line is.

Another word for gradient is 'slope'.

**Example**: find the gradient of the line drawn on the graph below




## *Intercept*

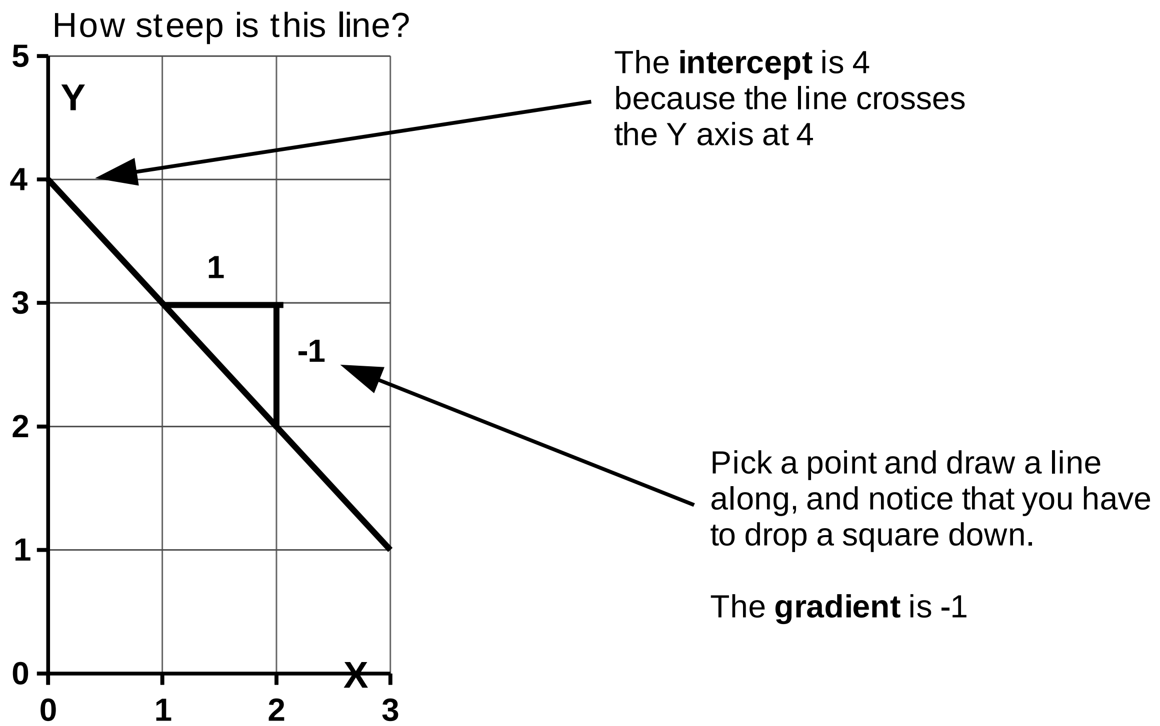
The intercept is the value of the Y coordinate when the line crosses the Y axis.

The intercept of the line above is 3 because the line crosses the Y axis at Y=3

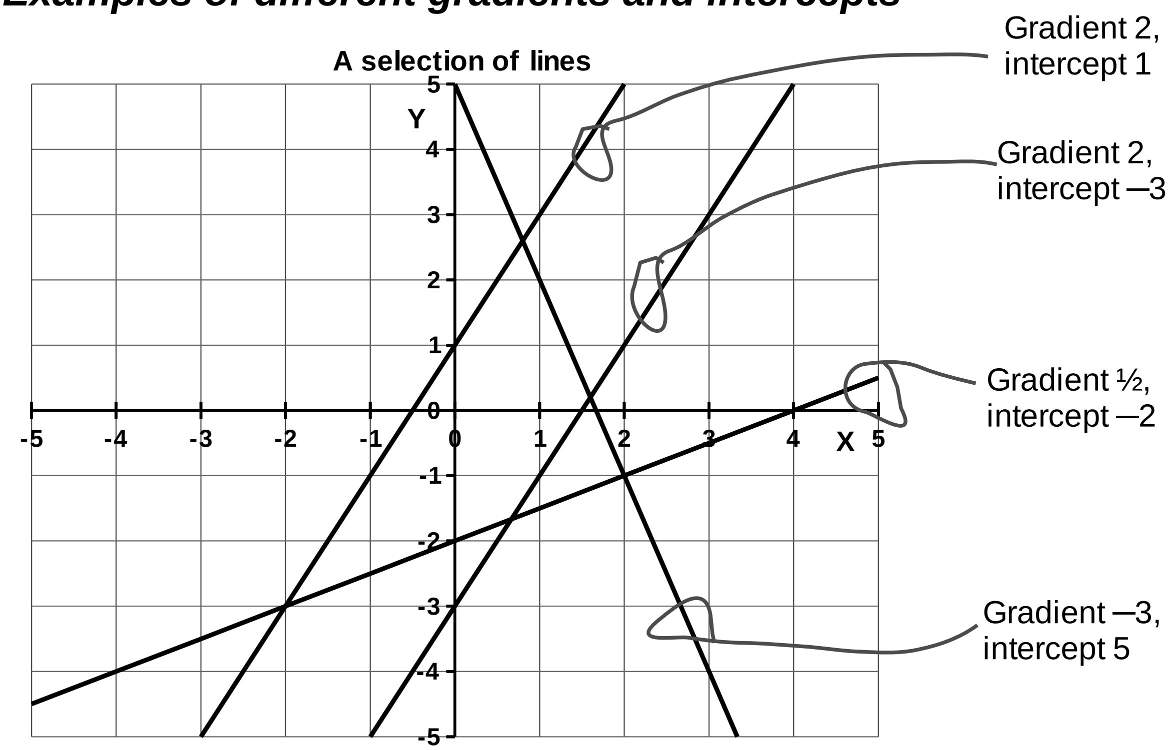
## *Negative gradients*

Lines can be 'downhill' so that an increase in X coordinate leads to a decrease in the Y coordinate.

**Example**: find the gradient or slope and the intercept of the line below




## *Examples of different gradients and intercepts*



## *Plotting from the equation*

You can use a formula or equation to make a table of values, and then you can plot a graph from the table of values.

**Example**: Plot the graph of $y=3x+2$ from $x=-3$ to $x=+3$

**Stage 1**: The formula $y=3x+2$ is like a function machine. It says to take a value of $x$, multiply it by 3 and add 2.

**Stage 2**: You run the formula machine on each value of $x$ in turn, –3, –2, –1, 0, 1, 2 and 3 to get the Y values you need to plot the graph

For X = –3, Y = 3 × –3 + 2 = –9 + 2 = –7, so (–3, –7) is a point on the graph

For X = –2, Y = 3 × –2 + 2 = –6 + 2 = –4, so (–2, –4) is a point on the graph

For X = –1, Y = 3 × –1 + 2 = –3 + 2 = –1, so (–1, –1) is a point on the graph

For X = 0, Y = 3 × 0 + 2 = 2, so (0, 2) is a point on the graph

For X = 1, Y = 3 × 1 + 2 = 3 + 2 = 5, so (1, 5) is a point on the graph

For X = 2, Y = 3 × 2 + 2 = 6 + 2 = 8, so (2, 8) is a point on the graph

For X = 3, Y = 3 × 3 + 2 = 9 + 2 = 11, so (3, 11) is a point on the graph

The working about above is very long winded, but make sure you follow the substitutions, especially the arithmetic with the negative numbers.

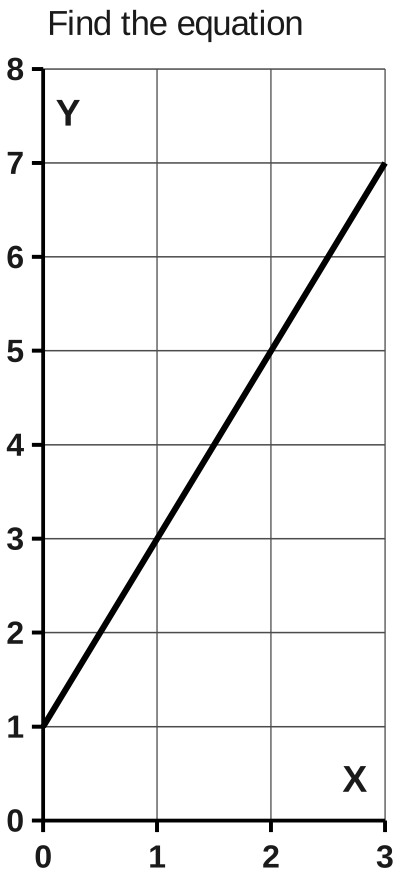
**Stage 3**: The easiest way to do this in an exam is to use a table

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| X | –3 | –2 | –1 | 0 | 1 | 2 | 3 |
| Y | –7 | –4 | –1 | 2 | 5 | 8 | 11 |

### *Finding the Equation from the graph*

If you can find the gradient and the intercept of the straight line, you can write the equation of the line down.

**Example**: find the equation of the graph below




**Stage 1**: write down the intercept, 1 in this case

**Stage 2**: find the gradient of the line, 2 in this case (see box above next to graph)

**Stage 3**: Write down the equation by using the gradient as the number in front of x, and the intercept as the number in front of Y

$y=2x+1$