Quadratic sequences and the n^{th} term formula

In this piece I am spelling out the algebra around quadratic sequences and the usual rules for finding the n^{th} term formula for a quadratic sequence. I have noticed that the text books and revision guides tend to skip the notation so I have made a point of defining the symbols I use carefully. I'm also presenting a 'top down' argument based on the general quadratic sequence with formula $y_n = an^2 + bn + c$.

Example of a quadratic sequence and notation

14, 25, 40, 59, 82 is an example of a quadratic sequence.

As you can see, the difference between one term and the next is not constant.

The so called n^{th} term formula for this sequence will have an algebraic term in n^2 as well as possibly a term in n and a constant term.

I'm using y_n to mean the value of the n^{th} term of the sequence. In the example, $y_1 = 14$, $y_2 = 25$, $y_3 = 40$ and so on. Some textbooks and Web sites might use t_n for the value of the n^{th} term of the sequence. Quite a few textbooks don't use any symbol at all for the values of the terms, just for the number of the term n.

It helps to clarify things if you make a table and add a row for the values of n:

n	1	2	3	4	5
y_n	14	25	40	59	82

The n^{th} term formula for the example above is $y_n = 2n^2 + 5n + 7$. You can split the expression into the square term, $2n^2$ and the linear term $5n + 7$.

The table of values above looks a bit like the table of values for plotting a quadratic function.

Take the table and add a couple of rows:

n	1		2		3		4		5
y_n	14		25		40		59		82
1st difference		11		15		19		23	
2nd difference			4		4		4		

The row labelled *1st difference* shows the differences between the terms of the quadratic sequence.

The row labelled *2nd difference* shows the differences between the 1st differences. As you can see these second differences are constant. That tells you that the original sequence is a quadratic sequence. If you had a cubic sequence with a term in n^3 then you would need to make three rows of differences before you saw a constant.

Using the differences to work out more values

You can use the first and second differences to calculate more values of the sequence by adding the second difference to the first difference to calculate the next first difference and then using that value to calculate the next term in the sequence.

For example suppose you had the quadratic sequence: 2, 6, 16, 32

Setting up a table and working out the first and second differences:

n	1		2		3		4		5		6		7
y_n	2		6		16		32						
1st diff		4		10		16							
2nd diff			6		6								

You can find y_5, the value of the sequence for n = 5 by

- Adding the 2nd difference onto the previous first difference: 16 + 6 = 22
- Adding the new 1st difference onto y_4, the previous sequence value (22 + 32 = 54)

n	1		2		3		4		5		6		7
y_n	2		6		16		32		***54***				
1st diff		4		10		16		***22***					
2nd diff			6		6		***6***						

Repeating this two stage procedure again will give you the value for y_6:

- 22 + 6 = 28 for 1st difference and
- 54 + 28 = 82 for y_6

n	1		2		3		4		5		6		7
y_n	2		6		16		32		54		***82***		
1st diff		4		10		16		22		***28***			
2nd diff			6		6		6		***6***				

And then you can calculate y_7 the sequence value for $n = 7$

- New 1st difference = 28 + 6 = 34
- So $y_7 = 82 + 34 = 116$

n	1		2		3		4		5		6		7
y_n	2		6		16		32		54		82		***116***
1st diff		4		10		16		22		28		***34***	
2nd diff			6		6		6		6		***6***		

See how you only had to use addition and subtraction to work out values of the quadratic? Before the invention of electronic calculators and computers, difference tables were used routinely to calculate the values of polynomials (quadratics, cubics and so on). Human beings (retired bank clerks often) sat in offices adding and subtracting and filling in the tables.

Next: doing some algebra with the most general quadratic sequence.

The general formula for a quadratic sequence

The most general quadratic sequence has the n^{th} term formula $y_n = an^2 + bn + c$ where y_n represents the value of the n^{th} term of the sequence. Very similar to the general quadratic.

The table below shows the values of the first 5 terms of the sequence, $n = 1$, $n = 2$... obtained by substituting the value for n into the general expression. I had to use a vertical table to provide enough space for the algebraic expressions, so the n values are in the first column rather than the first row.

The y_n column shows the substitution into the general expression for each value of n.

n	y_n	y_n simplified	1^{st} difference	2^{nd} difference
1	$a\times1^2+b\times1+c$	$a+b+c$		
			$3a+b$	
2	$a\times2^2+b\times2+c$	$4a+2b+c$		$2a$
			$5a+b$	
3	$a\times3^2+b\times3+c$	$9a+3b+c$		$2a$
			$7a+b$	
4	$a\times4^2+b\times4+c$	$16a+4b+c$		$2a$
			$9a+b$	
5	$a\times5^2+b\times5+c$	$25a+5b+c$		

As you can see in the 'y_n simplified' column, the value of each term is a linear combination of multiples of the coefficients a, b and c.

The table also includes the first and second differences in terms of the coefficients. The second differences are constant and all have the value $2a$ where a is the coefficient of n^2 in the n^{th} term formula.

That means you can work out the coefficient of n^2 for *any* quadratic sequence by finding the second difference and taking half of it.

Now $an^2+bn+c-an^2=bn+c$, so by subtracting off the an^2 term which you have found by taking half the second difference, you are left with a linear sequence. You can use your preferred method to find the n^{th} term formula for the linear sequence.

Finding the formula for the n^{th} term of a quadratic sequence

Finding the nth term formula for a quadratic sequence comes down to finding the values of the constants a, b and c in the general formula an^2+bn+c . Because you can find the coefficient of n^2 by taking half of the second difference, you can split off the an^2 term from a linear sequence $bn + c$.

Using the first example sequence 14, 25, 40, 59, 82, you start by writing out the terms you know in a table with the n values in the top row.

n	1	2	3	4	5
y_n	14	25	40	59	82

Working out the 1st and 2nd differences as before:

n	1		2		3		4		5
y_n	14		25		40		59		82
1st difference		11		15		19		23	
2nd difference			4		4		4		

You now know that the coefficient of n^2 must be half of the second difference, so $4 \div 2 = 2$ for this sequence.

The next step is to work out the values of $2n^2$ and writing them under the yn values in the table. The bottom row labelled 'linear' shows $y_n - 2n^2$.

n	1	2	3	4	5
y_n	14	25	40	59	82
$2n^2$	$2 \times 1^2 = 2$	$2 \times 2^2 = 8$	$2 \times 3^2 = 18$	$2 \times 4^2 = 32$	$2 \times 5^2 = 50$
linear	12	17	22	27	32

The linear sequence is 12, 17, 22, 27... which has difference 5 and 'zeroth term' 7. I counted one difference back from 12 to find the linear sequence value corresponding to $n = 0$.

n	0	1	2	3	4	5
y_n		14	25	40	59	82
$2n^2$		$2 \times 1^2 = 2$	$2 \times 2^2 = 8$	$2 \times 3^2 = 18$	$2 \times 4^2 = 32$	$2 \times 5^2 = 50$
linear	7	12	17	22	27	32

So the n^{th} term formula for the linear sequence is $5n + 7$, so b = 5 and c = 7 in the general formula an^2+bn+c . The nth term for this sequence is $y_n=2n^2+5n+7$.

Summary of the steps

1. Make a table with four rows, one for the term numbers n one for the term values y_n and one row each for the 1st and 2nd differences.

2. Check that the second difference is constant! If it is then the sequence is quadratic. Half the value of the 2nd difference gives you the value *a* in the an^2 term of the sequence.
3. Make another table with four rows but including an extra column for $n = 0$. Again the first row is the n values and the second is the term values y_n. Calculate the value of an^2 for each *n* and put these values in the third row. Put the difference $y_n - an^2$ in the fourth row.
4. The fourth row is a linear sequence. Count one term back to find the 'zeroth term' value y_0. Then write down the formula for the nth term.
5. The n^{th} term formula for the quadratic sequence is just an^2 plus the nth term formula for the linear sequence in step 4.

"Zeroth term" method for finding the nth term of a quadratic sequence

The method for finding the n^{th} term formula in the previous section does require writing out a number of tables. You can use an extended version of the 'zeroth term' method with quadratic sequences using some results from the 'general sequence' table.

I've reproduced the first few rows of the 'general sequence' table with an extra row for the 0^{th} term below:

n	y_n	y_n simplified	1st difference	2nd difference
0	$a\times0^2+b\times0+c$	c		
			$a+b$	
1	$a\times1^2+b\times1+c$	$a+b+c$		$2a$
			$3a+b$	
2	$a\times2^2+b\times2+c$	$4a+2b+c$		$2a$
			$5a+b$	
3	$a\times3^2+b\times3+c$	$9a+3b+c$		

As you can see, the value of the 0^{th} term is just the constant *c*, and half the 2nd difference gives you the value of *a*. The 1st difference between y_0 and y_1 is $a + b$, so as you know the value of *a* you can find the value of *b* by subtracting. This suggests a method for finding the values of the coefficients *a*, *b* and *c* in the n^{th} term formula $y_n = an^2 + bn + c$.

Step 1: Make a table including a column for $n = 0$ and find the 1st and 2nd differences

Step 2: Divide the 2nd difference by 2 to find the value of *a*.

Step 3: Find the value of the sequence y_0 for $n = 0$, the 'zeroth term', $y_0 = c$.

Step 4: The 1st difference $y_1 - y_0 = a + b$. So b = $y_1 - y_0 - a$.

Step 5: Now you have the values of the coefficients you can write out the n^{th} term formula.

Example: Find the nth term formula for the quadratic sequence: 4, 15, 32, 55

Step 1: Below is a table showing the 1st and second differences, and with a column for y_0.

n	0		1		2		3		4
y_n			4		15		32		55
1st diff				11		17		23	
2nd diff					6		6		

Step 2: $a = 6 \div 2 = 3$ so the coefficient of n_2 in the n^{th} term formula is 3.

Step 3: Find the value of the 'zeroth term' of the sequence. The table below shows the new entries in bold-italic.

n	0		1		2		3		4
y_n	***–1***		4		15		32		55
1st diff		***5***		11		17		23	
2nd diff			***6***		6		6		

Write the value of the 2nd difference in column 1, then as we are working backwards, *subtract* that difference from 11 to find the value of the 1st difference $y_1 - y_0$. Then work out the value of y_0 by *subtracting* the value of the 1st difference from y_1. In this case $4 - 5 = -1$ giving $c = -1$.

Step 4: The 1st difference $y_1 - y_0 = a + b$, so subtracting the value of a you found in step 1 (half the 2nd difference) from the 1st difference will give b. In this case $b = 5 - 3 = 2$.

Step 5: You have $a = 3$, $b = 2$ and $c = -1$ so the nth term formula is $y_n = 3n^2 + 2n - 1$.

It is probably a good idea to check this formula against one of the values you know, say using $n = 4$. $y_4 = 3 \times 4^2 + 2 \times 4 - 1 = 48 + 8 - 1 = 55$. So the formula works.

Try some yourself

Find a text book or Web page with some example questions about finding the n^{th} term of a quadratic sequence and see if you can work through these steps.

Find a range of examples, some with negative term values, some with negative first differences and some with negative second differences. You might find some sequences with fractional differences, e.g. the triangle numbers, 1, 3, 6, 10, 15.

Doing a wide range of questions deepens your knowledge of the topic. Familiarity tends to equal understanding.