Special angle values

The National Curriculum in England programmes of study state that students should;

"*know the exact values of sin θ and cos θ for θ = 0°, 30°, 45°, 60° and 90°; know the exact value of tan θ for θ = 0°, 30°, 45°, 60°*"

There is a small industry in making up clever schemes for students to remember these particular values. The word *exact* has to be taken to mean 'in surd form' where needed. So $\sin(60)=\frac{\sqrt{3}}{2}$ is exact and 0.8660254037844386467637231707529361834713 is approximate.

I think that it might be useful to see how you can calculate these values from the basic formulas using Pythagoras' result to find various lengths and trigonometry to form the exact values of each of the three trigonometry functions for the 30, 45 and 60 degree values.

The Values

The values you need to know are summarised in the table below:

	0°	30°	45°	60°	90°
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	(∞)

I've listed the values of sin(45) and cos(45) as $\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$ because having a no square roots on the bottom of the fraction is a common convention.

You will have noticed that the National Curriculum text is written *very carefully* at the cost of some repetition to not ask for the value of tan(90) because tan(90) is infinity. Asking your scientific calculator to find tan(90) will give a 'syntax error' or some other error message and you have to press the [AC] button to clear the calculator.

Values for 0º and 90º

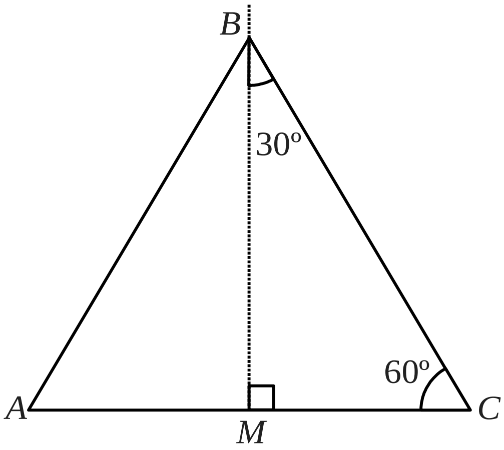
Imagine a very long thin right-angled triangle which has one side of zero length and the other two sides (say) 1 unit in length.

sin(0) = $\frac{opposite}{hypotenuse}=\frac{0}{1}=0$ and for the other angle sin(90) = $\frac{opposite}{hypotenuse}=\frac{1}{1}=1$

cos(0) = $\frac{adjacent}{hypotenuse}=\frac{1}{1}=1$ and for the other angle cos(90) = $\frac{adjacent}{hypotenuse}=\frac{0}{1}=0$

tan(0) = $\frac{\sin(0)}{\cos(0)}=\frac{0}{1}=0$

Half an equilateral triangle: values for 30º and 60º



Triangle *ABC* is equilateral so the angle *BCA* is 60°. Lets take the lengths of each of the sides *AB*, *BC* and *CA* to be 1 unit.

The triangle has an axis of symmetry that includes the line segment *BM* where *M* is the midpoint of *AC*. So angle *MBC* must be 30°, half of 60°.

Triangle *MBC* is a right angled triangle, so you can use Pythagoras' result to find the lengths of the sides. We said that the side *BC* has length 1 unit and this side is the hypotenuse of the triangle *MBC*, and the length of *MC* is $\frac{1}{2}$ a unit. You can find the length of *BM* using Pythagoras' result:

$BM^2 + MC^2 = BC^2$ so substituting values we know you get

$BM^2 + \left(\frac{1}{2}\right)^2 = 1^2$ so

$BM^2 = 1^2 - \left(\frac{1}{2}\right)^2 = 1^2 - \ 1-\frac{1}{4}=\frac{3}{4}$. Taking the square root gives $BM = \sqrt{\frac{3}{4}}$

Using some ideas from surds $\sqrt{\frac{3}{4}}=\frac{\sqrt{3}}{\sqrt{4}}=\frac{\sqrt{3}}{2}$ gives $BM = \frac{\sqrt{3}}{2}$.

Finding the sines

Remember that $\sin(A)=\frac{opposite}{hypotenuse}$

Looking at the 60° angle *MCB*, *BM* is the opposite side and *AC* is the hypotenuse

so $\sin(60)=\frac{\sqrt{3}}{2}\div 1=\frac{\sqrt{3}}{2}$ exactly.

Looking at the 30°, *CM* is the opposite side and *AC* is the hypotenuse again

so $\sin(30)=\frac{1}{2}\div 1=\frac{1}{2}$ exactly.

Finding the cosines

For the values of the cosines you could just use the identity $\cos(A) \equiv \sin(90 - A)$ to say $\cos(30) = \frac{\sqrt{3}}{2}$ and $\cos(60) = \frac{1}{2}$

Or you could use the trig formulas again;

$\cos(A) = \frac{adjacent}{hypotenuse}$

Looking at the 60° angle, *CM* = $\frac{1}{2}$ is the adjacent and *BC* = 1 the hypotenuse

which gives you cos(60)= $\frac{1}{2} \div 1 = \frac{1}{2}$ exactly.

Looking at the 30° angle, *BM*= $\frac{\sqrt{3}}{2}$ is the adjacent, and *BC* = 1 the hypotenuse

so cos(30) = $\frac{\sqrt{3}}{2} \div 1 = \frac{\sqrt{3}}{2}$ exactly.

Finding the tangents

$\tan(A) = \frac{opposite}{adjacent}$

For the 60° angle, *BM* = $\frac{\sqrt{3}}{2}$ is the opposite and *CM* = $\frac{1}{2}$ is the adjacent

which gives you tan(60) = $\frac{\sqrt{3}}{2} \div \frac{1}{2} = \frac{\sqrt{3}}{2} \times \frac{2}{1} = \sqrt{3}$.

For the 30° angle, the sides swap names of the sides of the triangle. *CM* = $\frac{1}{2}$ is now the opposite and *BM* = $\frac{\sqrt{3}}{2}$ becomes the adjacent.

Which results in tan(30) = $\frac{1}{2} \div \frac{\sqrt{3}}{2} = \frac{1}{2} \times \frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}}$. This makes sense because we swapped the names of the sides (what was the opposite becomes the adjacent and what was the adjacent becomes the opposite side) so the fraction turns upside down.

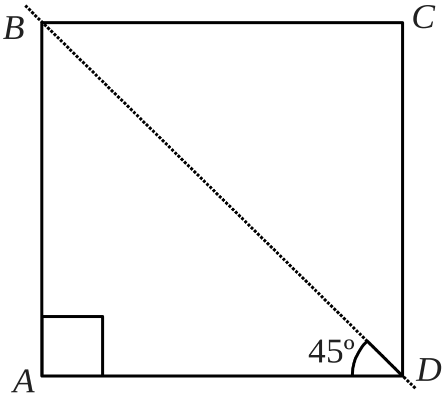
You can 'rationalise the denominator' so $\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$.

Some of the symmetries in the results might help you to remember the values.

- cos(30) is the same as sin(60) and
- sin(30) is the same as cos(60)
- tan(30) is the reciprocal of tan(60)

The next thing to do is to find the values of sine, cosine and tangent for 45° by slicing a square in half along the diagonal to make two right-angled triangles each with angles of 45°, 45° and the right angle.

Half a square: values for 45º



ABCD is a square of side 1 unit and *BD* is a diagonal.

Angle *BDA* is 45° and you can find the length of *BD* using Pythagoras' result in the right-angled triangle *BAD*;

BD is the hypotenuse so $BD^2 = AD^2 + DC^2 = 1^2 + 1^2 = 2$

Taking the square root, the length of *BD* = $\sqrt{2}$.

Sine of 45°

Looking at angle *ADB* in the right-angled triangle *BAD*, *BA* is the opposite side, so

$$\sin(45)=\frac{Opposite}{Hypotenuse}=\frac{BA}{BD}=\frac{1}{\sqrt{2}}\ .$$

You might want to 'rationalise the denominator' so there isn't a square root on the bottom of the fraction: $\frac{1}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}=\frac{\sqrt{2}}{2}$

Cosine of 45°

Looking at the same triangle: see how the adjacent is the same length as the opposite, so for 45° the sine and cosine have the same value!

Going through the arithmetic with *BD* as the hypotenuse and *AD* as the adjacent:

$$\cos(45)=\frac{Adjacent}{Hypotenuse}=\frac{AD}{BD}=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$$

Tangent of 45°

As the sine and cosine of 45° have the same values, the tangent of 45° must be 1 exactly because $\tan(A)=\frac{\sin(A)}{\cos(A)}$.

Or you can use trigonometry in triangle BAD with BA as opposite and AD as adjacent;

$$\tan(45)=\frac{Opposite}{Adjacent}=\frac{BA}{AD}=\frac{1}{1}=1 \text{ exactly.}$$