Probability rules and tools

Fractions and decimals refresher

Work out each calculation without using a calculator.

1) $1 - 0.35$
2) $1 - (0.2 + 0.15 \times 3)$
3) $\frac{3}{4} \times \frac{2}{5}$
4) $\frac{1}{3} + \frac{2}{7}$
5) $1 - \frac{5}{6}$ (Remember that $1 = \frac{6}{6}$)
6) $\frac{3}{8} \times \frac{5}{7} + \frac{5}{8} \times \frac{2}{7}$ (BODMAS applies to fractions)
7) $\left(\frac{5}{6}\right)^3$
8) Find two-thirds of 480
9) $\frac{3}{7} \times 1400$
10) A bag contains 45 yellow counters and 25 blue counters only.
What fraction of the counters in the bag are yellow?

Check you get the same answers using your calculator.

Probability is a fraction

Probability is a fraction, the fraction of time something happens.

Toss a coin. You will see a head half the time.

The probability of tossing a coin and seeing a head is $\frac{1}{2}$.

If something always happens, the probability is 1 (it happens 100% of the time).

If something never happens, the probability is 0 (it happens 0% of the time).

So probability is *always* a fraction (which can be a decimal) between 0 and 1.

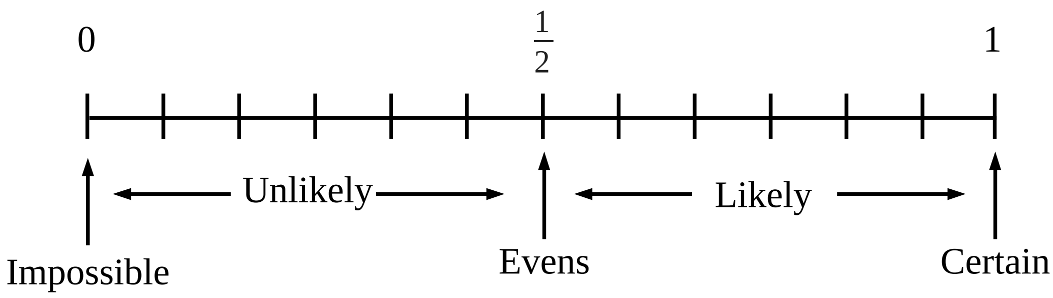
Using a fraction to represent probability means that you can make predictions about what might happen when you try something.

Probability ideas started with gambling and games of chance but now underlie important industries like insurance, medical drug research and so-called 'machine learning' in IT.

People roll dice to get a score to use in games. The singular of dice is die (like mice and mouse) but I've noticed that the exam questions use the word dice if there is one or several so I'm doing the same.

Probability scale

Below is a probability scale like a ruler.



This particular scale is divided into 12 equal sections.

You can use any scale, decimals or fractions, that fit the situation you have. If you have (say) a question about 8 marbles in a bag it would make sense to divide the probability scale into eight equal sections.

Any outcome with a probability greater than zero but less than $\frac{1}{2}$ can be described as *unlikely*.

And any outcome with a probability greater than $\frac{1}{2}$ but less than 1 can be described as *likely*.

Evens is used for outcomes with probability half. *Certain* and *impossible* are used to describe probabilities of 1 or 0.

Probability formula

Simple things like rolling a dice or tossing a coin have equally likely outcomes because of their symmetry.

The probability of rolling a 5 on an ordinary dice is 1 way out of 6 possible scores.

The probability fraction is $\frac{1}{6}$.

The formula below can help work out probabilities.

$$Probability = \frac{Number\ of\ ways\ you\ get\ desired\ outcome}{Total\ number\ of\ equally\ likely\ outcomes}$$

Example: suppose I have 5 red marbles and 3 yellow marbles in a bag.
I pick one marble at random from the bag and note the colour.
Work out the probability that the marble I picked was yellow...

Ans: Each marble of whatever colour has an equal chance of being chosen.

There are 8 marbles in total in the bag.

Three of the marbles are yellow, so using the probability formula...

$$Probability = \frac{Number\ of\ ways\ you\ get\ desired\ outcome}{Total\ number\ of\ equally\ likely\ outcomes} = \frac{3}{8}$$

Experimental probabilities

Important things in life are generally much more complicated than tossing coins, rolling dice or spinning spinners.

People do surveys and collect the results and then calculate probabilities based on the result.

Example: Georg plants 250 sweet pea seedlings.

75 of the seedlings grow into plants that produce yellow flowers.

Calculate an *estimate* of the probability of getting a yellow flower.

Ans: You apply the probability formula to the results you get so the probability of planting a seedling and getting a yellow flower is $\frac{75}{250}=0.3$.

The word **estimate** is used in the question because the probability has been calculated from actual results.

Try the experiment again and the results will be a bit different.

As you increase the number of seeds you plant, the *percentage* or *proportional* variation in the results will get smaller.

The word **estimate** is <u>not</u> telling you to round the numbers in this context.

They use the word estimate to allow for the fact that experimental probabilities vary each time you try the experiment.

Some textbooks and revision guides might use **relative frequency** for experimental probability.

Expected frequency

$e.f. = probability \times number\ of\ trials$

You can use probability to make predictions.

Toss a dice 600 times.

How many times would you expect to see a score higher than four?

The score can be a 5 or a 6.

The probability formula says the probability is $\frac{2}{6}$ so about two sixths of the time you would expect a score higher than four.

$\frac{2}{6}$ of 600 is 200.

So the **expected frequency** of a score higher than four is 200.

Trials, outcomes and events

Textbooks videos and revision guides use words to describe probability questions.

Trial or **experiment**: the thing you do like rolling a dice or tossing a coin

Outcome: what you see e.g. the score on the dice or a head/tail on a coin

Event: seems to be used like outcome, but sometimes used for combinations of outcomes.
The trial could be rolling a dice an event could be seeing a score that is a square number.

The outcomes that are consistent with the event 'seeing a square number' would be 1 and 4.

Listing all outcomes systematically

Example: Suppose a restaurant has three choices of starter and four choices of main course to make a meal.

Starters: Olives, Salad, Prawns

Main courses: Chicken, Fish, Vegetarian, Lamb

List all the possible meals that could be chosen.

Ans: First work out how many different combinations there are
$3 \times 4 = 12$ so there are 12 different meals that could be ordered.

Then it is best to work systematically to list the 12 combinations...

List the Olives with all four mains: OC, OF, OV, OL

then list the Salad with all four mains: SC, SF, SV, SL

finally list the Prawns with all four mains: PC, PF, PV, PL

Check you have the right number of combinations.

Mutually exclusive events OR rule (or means add)

Suppose you roll a fair dice.

The outcome could be 1 or 2 or 3 or 4 or 5 or 6.

If you get a score of 4, you can't also get a score of 3. Seeing the 4 outcome *excludes* the 3 outcome.

The set of possible dice scores {1, 2, 3, 4, 5, 6} is called **mutually exclusive** because getting any one of them excludes the others.

Suppose you want the probability of seeing a score of 2 *or* 3 *or* 4. Because these outcomes are mutually exclusive, you can just **add the probabilities** $\frac{1}{6}+\frac{1}{6}+\frac{1}{6}=\frac{3}{6}=\frac{1}{2}$

Probabilities add to one

In a trial something has to happen so the total of the probabilities of all the outcomes must be 1.

There is always a set of mutually exclusive outcomes.

At the very least there are the two mutually exclusive outcomes {A happens, A does not happen}.

Example 1: Fred rolls a fair dice. Work out the probability that he does <u>not</u> see a score of 5.

Ans: Probability of seeing a score of 5 is $\frac{1}{6}$ so probability of not seeing a score of 5 must be

$$1-\frac{1}{6}=\frac{6}{6}-\frac{1}{6}=\frac{5}{6}$$

Example 2: Ethan rolls a biased dice. The probabilities of various scores are shown in the table below...

Score	1	2	3	4	5	6
Probability	0.1	0.2	0.2	0.2	x	$2x$

Find the probability of a score of 6

Ans: Probabilities must add to 1 so 0.1 + 0.2 + 0.2 + 0.2 + 3x = 1 which simplifies to 0.7 + 3x = 1.

Solving for x gives x = 0.1.

Substituting into 2x gives probability of rolling a 6 of 0.2.

Independent events AND rule (and means multiply)

Suppose you decide to make a game where people toss a coin *and* roll a die to decide what their next move is.

The tossing of the coin is independent of the rolling of the die.

Whatever the result of the coin toss, all the outcomes of the die are still equally likely.

Probability of head *and* four = probability of head on coin × probability of four on dice

Probability of head *and* four = $\frac{1}{2}\times\frac{1}{6}=\frac{1}{12}$

You could systematically list the 12 outcomes of tossing a coin and rolling a dice (H1, H2, and so on) and then use the probability formula (1 way of getting a head and a four and 12 equally likely outcomes).

The word 'and' suggests multiplying the probabilities.

When you multiply fractions less than one, the answer is *smaller* than the fractions you started with.

It is less likely to see two separate outcomes together than each of those outcomes on their own.

Example: You have three red marbles and two blue marbles in a bag.

You pick one marble at random write the colour down and put it back in the bag.

You pick another marble from the bag and write the colour down.

Work out the probability that you get two blue marbles

Ans: Probability of picking a blue marble is $\frac{2}{5}$ from the probability formula

Probability of picking a blue marble *and* another blue marble is $\frac{2}{5}\times\frac{2}{5}=\frac{4}{25}$ by the and rule.

Possibility Space diagram (table)

A possibility space diagram is another way of showing the outcomes for two independent trials.

The 'diagram' (really a table) shows all the possible outcomes of two trials so you can use the probability formula to find the probabilities of various events.

Example: Draw a possibility space diagram showing all the possible outcomes of tossing a coin and rolling a dice.

Ans: The 'diagram' is a two way table with each cell showing a possible outcome.

	1	2	3	4	5	6
H	H1	H2	H3	H4	H5	H6
T	T1	T2	T3	T4	T5	T6

Example: Find probability of rolling a dice and tossing a coin and seeing a head and an even number

Ans: Three outcomes H2, H4 and H6 fit the required event and there are 12 equally likely outcomes so using the probability formula gives $\frac{3}{12}=\frac{1}{4}$ as the probability.

Another example is where you have two events that generate scores and you decide to add them.

Example: Two dice are rolled and their scores added together to make a total.

Draw a possibility space diagram showing the possible totals.

Ans: The lowest possible score is 2 and the highest possible score is 12.

There are 6 × 6 = 36 equally likely outcomes so some possible scores must occur more than once.

Below is the table showing 36 equally likely outcomes.

See how some totals are repeated in the table.

There are six different ways of getting a score of 7, but only one way to get a score of 2.

+	1	2	3	4	5	6
1	2	3	4	<u>5</u>	<u>6</u>	<u>7</u>
2	3	4	<u>5</u>	<u>6</u>	<u>7</u>	<u>8</u>
3	4	<u>5</u>	<u>6</u>	<u>7</u>	<u>8</u>	<u>9</u>
4	<u>5</u>	<u>6</u>	<u>7</u>	<u>8</u>	<u>9</u>	10
5	<u>6</u>	<u>7</u>	<u>8</u>	<u>9</u>	10	11
6	<u>7</u>	<u>8</u>	<u>9</u>	10	11	12

You can use the possibility space diagram to find probabilities.

Example: Work out the probability of a total score that is greater than 4 but less than 10.

Ans: Total scores that satisfy the requirement in the question are 5, 6, 7, 8 and 9

There are 24 outcomes that fit in the table (underlined) so the probability is $\frac{24}{36}=\frac{2}{3}$ using the probability formula.

Tree diagrams: with replacement

A tree diagram is another way of describing a situation with combined events.

The diagram helps you see which paths lead to an event.

A tree diagram is useful when the probabilities of the combined outcomes are not equal.

Example: A bag has 3 blue balls and 5 yellow balls.

You pick a ball from the bag at random, note the colour and then *return the ball to the bag*.

You pick a second ball from the bag and note the colour.

a) Draw a tree diagram to represent the probabilities of the various combined outcomes.

b) Work out the probability of obtaining two blue balls in a row.

c) Work out the probability of obtaining a ball of each colour.

Ans: The first step is to sort out the probabilities.

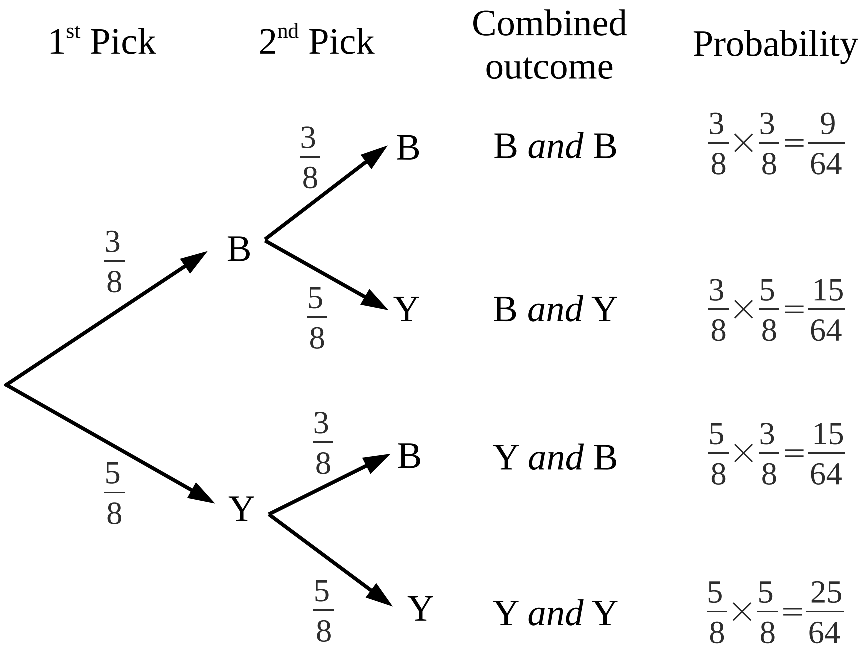
Before drawing the tree diagram, you need the probabilities of picking a blue ball and a yellow ball. Using the probability formula with 8 balls in the bag:

Probability of picking a blue ball: $\frac{3}{8}$

Probability of picking a yellow ball: $\frac{5}{8}$

Because you put the first ball you picked back in the bag before picking the second, these probabilities are the same for both first and second pick.

Ans for part a): The tree diagram is drawn below.



Notes on the tree diagram.

The left hand pair of lines or branches represents the outcomes for the first ball you pick.

The two sets of two branches for the second pick represent the outcomes for the second pick. There have to be two sets of branches.

The upper pair of branches show the outcomes given that the first ball was Blue. The lower pair of branches show the outcomes given that the first ball was yellow.

Notice how the probability of each branch is written along the line and the outcome for each branch is written at the end of each line (using a letter like B or Y to save space).

The **probabilities for each pair of branches add up to 1** because the two outcomes are mutually exclusive.

Ans b): The route through the tree diagram for two blue balls in a row is Start → B → B.

You find the probability of B and B by **multiplying the probabilities along the branches** that lead to the B and B outcome: $\frac{3}{8} \times \frac{3}{8} = \frac{9}{64}$

Ans c): To end up with one ball of each colour you don't care which order the colours come in.

Outcome B and Y *or* outcome Y and B will get you one of each colour.

There are two routes through the tree diagram, Start → B → Y or Start → Y → B.

Or means to add the probabilities of these two routes so the total probability of ending up with a ball of each colour is $\frac{3}{8} \times \frac{5}{8} + \frac{5}{8} \times \frac{3}{8} = \frac{30}{64}$

Tree diagrams: without replacement

Suppose we change the game from the last section:

Example: A bag has 3 blue balls and 5 yellow balls.

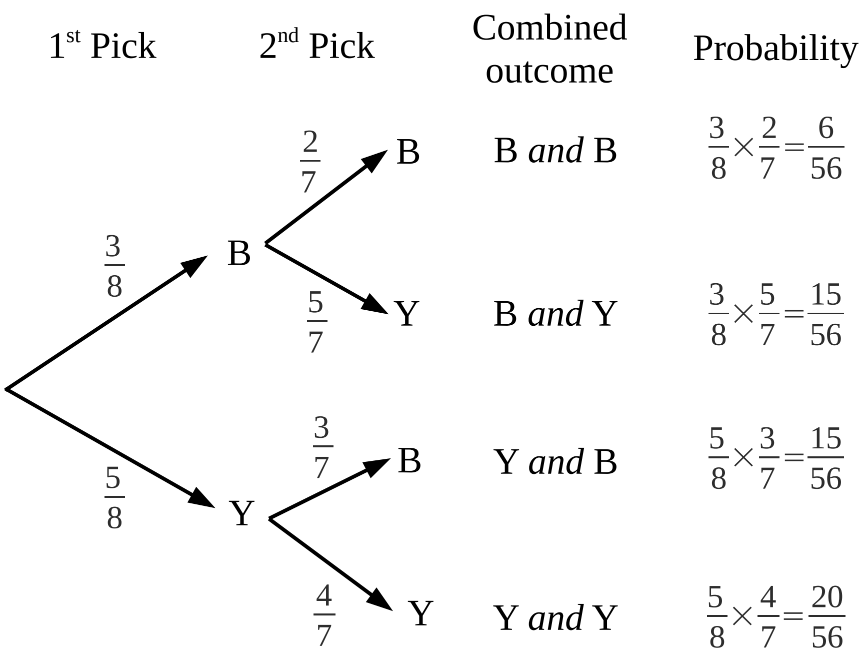
You pick a ball from the bag at random, note the colour *and don't replace the ball in the bag*.

You pick a second ball from the bag and note the colour.

a) Draw a tree diagram to represent the probabilities of the various combined outcomes.

b) Work out the probability of getting *at least* one yellow ball.

Ans a): The tree diagram is shown below with the combined outcomes and the probability calculation for each outcome.



The probabilities for the second pick change because you don't put the ball back after the first pick.

When you pick the first ball, there are only 7 balls left so the second pick probabilities have denominator 7.

If you picked a blue ball first, there would only be 2 blue balls left for the second pick, so using the probability formula the probability of picking another blue ball is $\frac{2}{7}$.

But there are still 5 yellow balls left so the probability of picking a yellow ball is $\frac{5}{7}$.

If you picked a yellow ball first then there are 3 blue balls and 4 yellow balls left for the second pick so the probabilities change as the diagram shows.

Ans b): *At least* one yellow ball means that you can have one yellow ball or two yellow balls.

Three of the outcomes in the tree diagram are relevant.

(B and Y) *or* (Y and B) *or* (Y and Y).

You just add up the three probabilities $\frac{15}{56}+\frac{15}{56}+\frac{20}{56}=\frac{50}{56}$

(Another way would be to say at least one yellow is the same as 1 – probability of B *and* B).

Tree diagrams: 'everyday' situations

These work the same way as without replacement questions because the second 'pick' probabilities can be different to the first 'pick'.

You would normally have to work out some of the probabilities given the information in the question.

Example: Fred can take the train (T) or walk to work (W). He can be early (E) or late (L).

The probability that Fred takes the train on any given day is 0.6

If Fred takes the train then the probability that he is late is 0.2

If Fred walks then the probability that he is late is 0.7

a) Draw a fully labelled tree diagram based on this information.

b) Work out the probability that Fred will be late on any given day.

Ans a): The first thing to do is to list the possible combined outcomes.

Fred can...

Take the train and be early (T and E)

Take the train and be late (T and L)

Walk and be early (W and E)

Walk and be late (W and L)

Next you need some probabilities:

The probability that he takes the train is given as 0.6 so the probability that he walks is 1 – 0.6 = 0.4 and that gives you the 1st Pick branches.

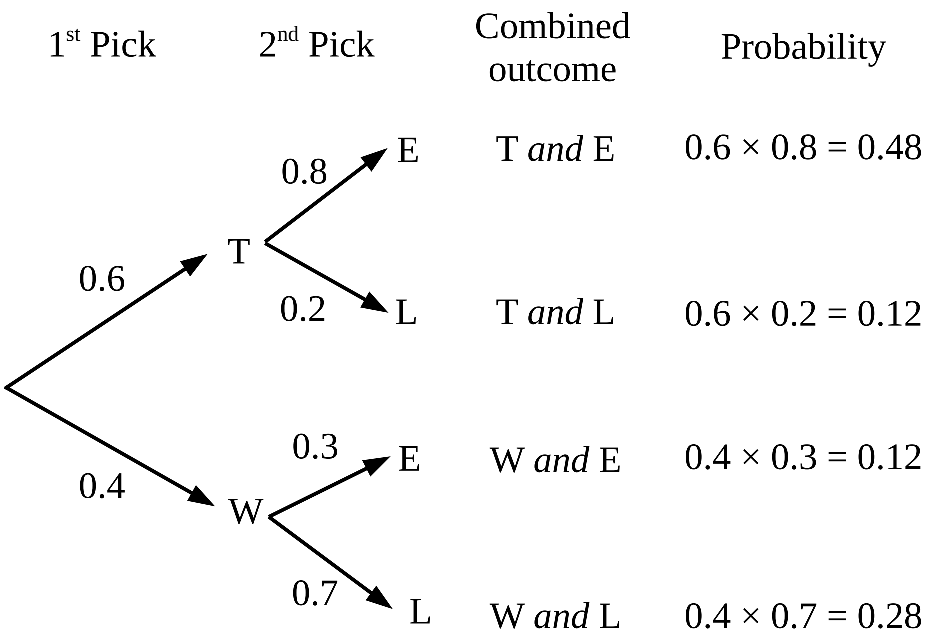
If he takes the train then the probability that he is late is 0.2 so the probability that he is early is 1-0.2 = 0.8.

You can draw the 2nd Pick upper branches.

If he walks then the probability that he is late is 0.7 so the probability that he is early is 0.3.

You can draw the 2nd Pick lower branches.

The completed tree diagram is below.



Ans b): There are two combined outcomes that end up with Fred being late:

(T *and* L) *or* (W *and* L). You can add the probabilities of those outcomes as they are mutually exclusive so total probability that Fred is late is 0.12 + 0.12 = 0.24

Two way tables and probability

A two way table presents the number of people or things broken down by two different categories.

You can use the frequencies in each cell of the table to work out probabilities when a person or thing is chosen at random.

Example: Each pupil in a school can choose an activity: either Outdoor or Team Games or Athletics.

The table below shows the number of pupils in year 8 and in year 9 that chose each activity.

	Outdoor	Team Games	Athletics	Total
Year 8	37	105	72	240
Year 9	43	85	98	200
Total	80	190	170	440

Use the information in the table to answer the following questions

a) What is the probability that a pupil chosen at random will pick Team Games?
b) What is the probability that a pupil chosen at random is in Year 8 *and* chooses Athletics?
c) Given that a pupil chose Outdoor, what is the probability that they are in Year 9?

Ans

You can use the probability formula to write down the answers to these questions, but you have to identify the top and bottom of the fraction correctly.

a) The pupil is chosen from the whole table so the bottom of the probability fraction is 440 . You don't care what year the pupil is in so the top of the fraction is the total number of pupils who chose Team Games which is 190.
The probability is $\frac{190}{440}=\frac{19}{44}$ that a pupil chose Team Games.

b) Again the pupil is chosen from the whole table so the bottom of the probability fraction is 440.
You only want the pupils in year 8 who chose Athletics, so 72 must be the top.
The probability is $\frac{72}{440}=\frac{9}{55}$ that the pupil is in year 8 and chose athletics.

c) You are only concerned with the pupils who chose Outdoor, so the bottom of the probability fraction must be 80.
The top of the probability fraction must be the number of pupils in year 9 which is 43.
The probability fraction is $\frac{43}{80}$.

The p(A) notation

A few of the questions might use a more abstract notation for the probability of an outcome.

P(R) is a short hand way of saying “the probability of outcome R in the trial”.

Example: a bag contains at least 50 marbles that can only be red (R) or yellow (Y)

P(R) is $\frac{2}{3}$. Write down the smallest possible number of yellow marbles.

Ans: The probability has denominator (bottom) 3 so the total number of marbles must be a multiple of 3.

The smallest multiple of 3 that is larger than 50 is 51 (3×17) so there must be 51 marbles.

P(Y) = 1 – P(R) = $\frac{1}{3}$ and $\frac{1}{3}$ of 51 is 17 so the smallest number of yellow marbles must be 17.

Venn diagrams, sets and probabilities

A *tiny* number of questions might use more abstract concepts like sets and Venn diagrams.

Venn diagrams can be used to sort out information anywhere not just probability.

A set in maths is a collection of things

Example: {square, rectangle, trapezium, parallelogram, kite, arrowhead, rhombus, scalene quadrilateral} is the set of types quadrilateral.

The set has elements (each shape) and you can describe the set by listing the elements.

You can also describe the set by using a definition: {polygons with four sides}

Sets can have subsets

Example: The set {1, 2, 3, 4, 5, 6} has a lot of subsets including the empty set {}, {1, 4} and {2, 3, 5}.

How would you describe {2, 3, 5} using a definition?

Ans: {The first three prime numbers}

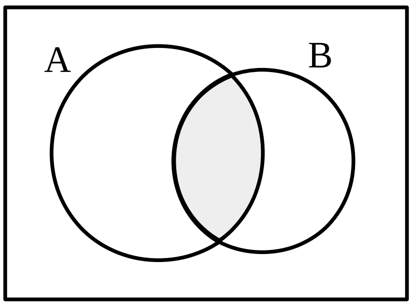
Intersection, union and complement

Being able to recognise these terms will be useful for a tiny number of questions.

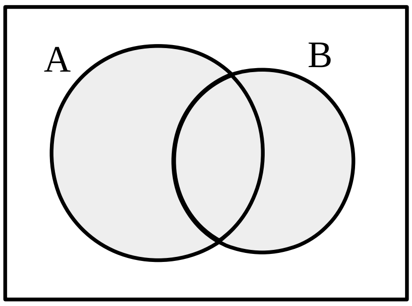
In the table below A and B are two sets

Symbols	Meaning	Example (scores on a dice)
$A \cup B$	The **Union** of set A and Set B is a new set that has all the elements of both sets Elements that are in both *A* and *B* are just listed once.	*A* = {1, 2, 4, 6} *B* = {2, 3, 5} $A \cup B = \{1, 2, 4, 6, 3, 5\}$
$A \cap B$	The **Intersection** of set A and set B is a new set that has elements that are in both *A* and in *B*.	A = {1, 2, 3, 4} B = {1, 3, 5} $A \cap B = \{1, 3\}$
A'	The **Complement** of set A is the set of things that are *not* in A.	A = {1, 6} Within the 'Universal Set' of scores on a dice, A' = {2, 3, 4, 5}
ξ	The **Universal Set** is the context or ground for the sets you are working with. It has the Greek letter epsilon as a symbol. See Venn Diagrams below.	The examples in this column are all based on the set of scores you find on an ordinary game dice. So for these examples, ξ = {1, 2, 3, 4, 5, 6}.
$\emptyset$	The **Empty Set** is the set with no members.	If If ξ = {1, 2, 3, 4, 5, 6} then the set { scores > 7 } = $\emptyset$

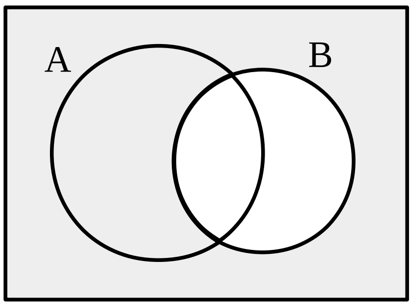
You can visualise these notations using Venn diagrams...



$A \cap B$



$(A \cup B)$



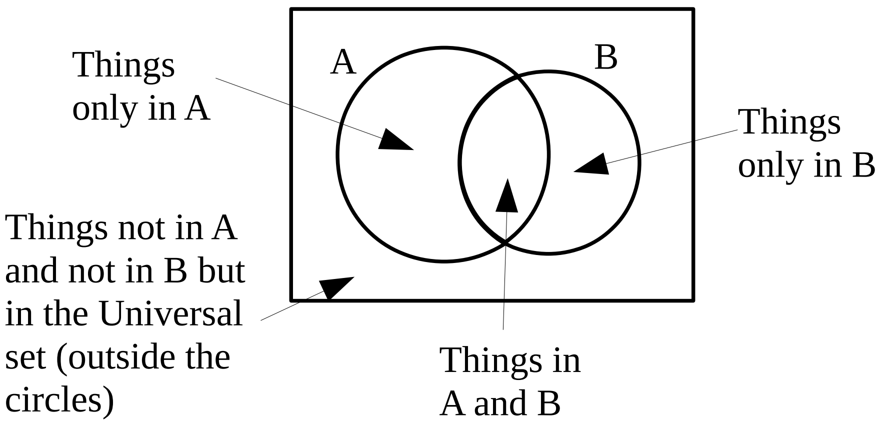
B'

Venn Diagrams

In a Venn diagram circles represent sets and the rectangle represents the Universal Set (the context of the question if you like).

The Venn diagram below shows two sets A and B with an intersection.

The diagram has four distinct *regions* described in the labels.



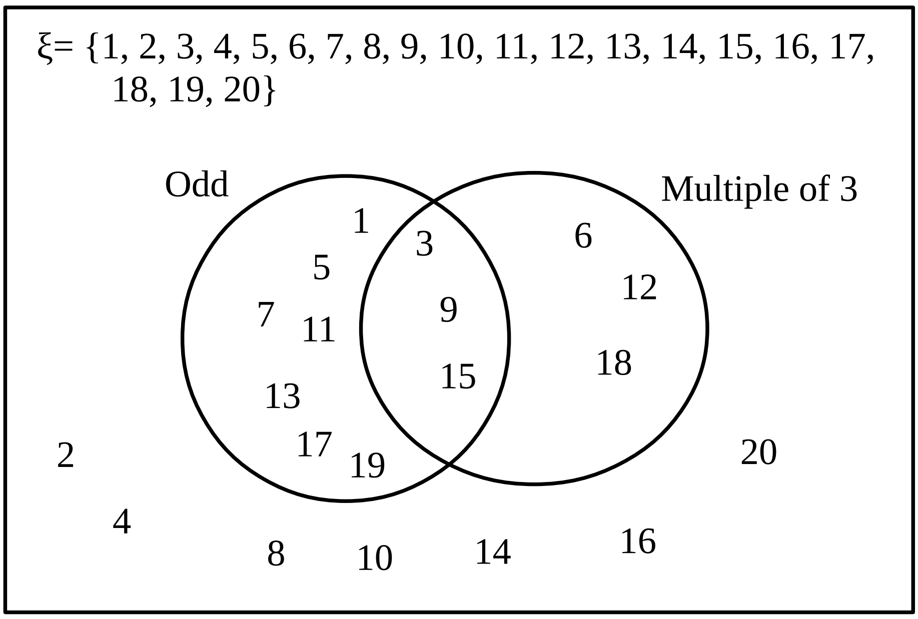
Example: Suppose you have a set of cards with the whole numbers 1 to 20 written on them so each card just has one number.

Draw a Venn diagram to classify the numbers according to if they are a multiple of three or an odd number.

Ans: There are four possibilities for any given number/card

- A number is odd and not a multiple of three
- A number is a multiple of three and not odd (therefore even)
- A number is both a multiple of three and odd
- A number is not a multiple of three and is not odd (therefore even)

Below is the Venn diagram that shows how the cards are classified into the four categories.



Using a Venn diagram to find a probability

Example: Suppose you have a set of cards with the whole numbers 1 to 20 written on them so each card just has one number.

You shuffle the 20 cards well and pick a card at random

Write down the probability of picking a card that is a multiple of 3 or an odd number

Ans: Count the number of cards within the circles.

You only count the cards in the intersection once!

P(odd or multiple of 3) = $\frac{13}{20}$

The way the Venn diagram classifies the elements of each set into the different regions allows you to work out what the numerator and denominator of the probability is.

Example: You shuffle the pack again and pick another card.

The card is a multiple of 3

Write down the probability that the card is odd.

Ans: Looking in the Multiples of Three circle in the Venn diagram there are 6 multiples of three so the denominator of the probability fraction is 6.

Looking at the intersection of the two sets, there are 3 numbers that are both multiples of3 and odd. That tells you that the numerator of the probability fraction is 3.

So the probability fraction is $\frac{3}{6}=\frac{1}{2}$

Probabilities of events that are not mutually exclusive

Recap of mutually exclusive events. The OR rule for mutually exclusive events says that you can add the probabilities of each event to get the overall probability.

Example: Roll an ordinary dice.

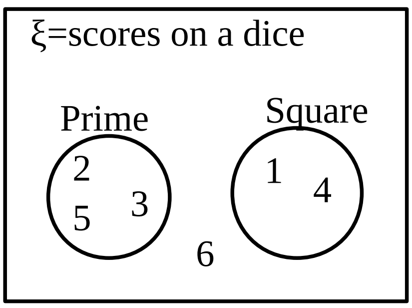
Work out P(prime OR square).

Ans: Prime scores are 2, 3 and 5. The probability of a prime score is $\frac{3}{6}$

Square number scores are 1 and 4. The probability of a square number score is $\frac{2}{6}$

As these outcomes are mutually exclusive, P(prime OR square) = $\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$

A Venn diagram of the possible scores would look like this...



As you can see there is no intersection, these events are mutually exclusive.

Events that are not mutually exclusive: You can extend the OR rule to work with events that are not mutually exclusive...

Example: Roll an ordinary dice

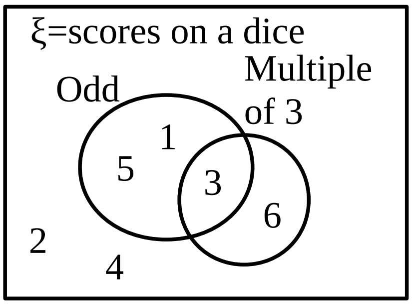
Work out P(odd OR multiple of 3)

Ans: Odd scores on a dice are 1, 3, 5 so P(odd) = $\frac{3}{6}$

Scores that are multiples of 3 on a dice are 3, 6 so P(multiple of 3) = $\frac{2}{6}$

The score 3 satisfies both conditions and so we *can't* add the probabilities directly.

A Venn diagram will help visualise what is going on...



If you just add the probabilities of the separate outcomes you will have counted the 3 twice.

The score 3 is in the intersection of both sets of scores, so 3 is a multiple of 3 *and* odd.

P(odd AND multiple of 3)= $\frac{1}{6}$

So you can add the probabilities of the outcomes then *subtract* the probability of the score being both a multiple of three *and* odd so you don't count it twice:

P(odd OR multiple of 3) = P(odd) + P(multiple of 3) – P(odd *and* multiple of 3)

P(odd OR multiple of 3) = $\frac{3}{6}+\frac{2}{6}-\frac{1}{6}=\frac{4}{6}$

A quick look at the Venn diagram shows that there are four scores in the circles representing the two events.

A formula to learn is:

P(A OR B) = P(A) + P(B) – P(A AND B)

Practice questions

Now you need to find a large collection of probability questions with answers or better complete worked solutions.

Try as many as you can and refer back to these rules and tools to check what rules you are using.

Challenge: can you summarise this whole handout on one side of A4?